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CONICAL SHELL VIBRATIONS

by Daniel H. Platus

Aerospace Research Associates, Inc.

West Covina, Calif.

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SUMMARY

An analytical method is presented for predicting the natural vibration frequencies and mode shapes of thin-walled conical shells rigidly fixed at one end and free at the other. Both extensional and inextensional theories are examined separately and a simplifying approximation is introduced for combining the results for the general case. The extensional theory is treated by using a Rayleigh-Ritz procedure with assumed mode shapes in the form of polynomial expressions in the axial shell coordinate, selected to satisfy the fixed-end displacement conditions. The natural frequencies and mode shapes are computed by using an IBM 7090 machine program developed for the extensional case. Results obtained by the approximate method show good agreement with experimental values obtained from actual vibration tests of thin-walled conical shells.

INTRODUCTION

Among the earliest treatments of conical shell vibrations was an analysis by Strutt (ref. 1) of the inextensional vibrations of the fixed-end, free-end conical shell. It will be shown in the present analysis that inextensional theory alone cannot accommodate the boundary conditions of a completely restrained edge. Consequently, the actual geometry investigated by Strutt was that of a free-end conical shell with a semiflexible attachment. The inadequacy of inextensional theory was also borne out by Van Urk and Hut (ref. 2) in their experiments to verify the applicability of Strutt's formula. Frequencies measured by vibrating thin aluminum shells differed from Strutt's predictions by a factor of 2 to 3.

The general problem, including extension of the middle surface of the shell, was investigated by Federhofer (ref. 3), who derived a frequency expression for vibrations of a truncated conical shell rigidly restrained at one edge and partially restrained at the other. The calculation was based on an energy method of Rayleigh, with assumed mode shapes in the form of simple power series. The mode shapes were selected to give a nonsingular solution for the complete conical shell restrained at its base and at its apex. This configuration was

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numerically evaluated as a special case of the problem. More recently, Grigolyuk (ref. 4) solved a similar problem using mode shapes which approximate a simply supported edge; that is, the radial and tangential deflections vanish at the edges, although the shell is unrestrained in the axial direction. Herrmann and Mirsky (ref. 5) investigated the same problem using a Rayleigh-Ritz method with sinusoidal mode shapes which exactly satisfy the equations of motion for a cylindrical shell in the limit of zero half-angle. Therefore, it would be expected that their results would apply only for cones with small divergence half-angles. Goldberg et al. treated the fixed-free cone, considering first only the axisymmetric (breathing) modes of vibration (ref. 6) and later the unsymmetric modes (refs. 7 and 8). The method of references 6 to 8 involves a technique of numerically integrating the differential equations of motion and is applicable to an arbitrary set of end conditions. Shulman (ref. 9) solved the simply supported case by a general energy method which, with enough terms included in the mode shapes, would probably give accurate results.

Recently, Saunders et al. treated two cases of the free-end cone. In the first case (ref. 10), the cone is attached at the smaller end to a segment of a spherical shell and is free at the other end. Only inextensional theory is considered and the calculated frequencies are found to agree closely with experimentally obtained values. In the second case (ref. 11), the cone is assumed to be completely restrained at one end and free at the other. Both extensional and inextensional theories are considered using a Rayleigh-Ritz procedure. Assumed mode shapes are represented in the form of simple polynomial expressions and the assumption is made (with no apparent justification) that two of the three principal extensional strains are zero. The resulting mode shapes, in view of this assumption, do not satisfy the fixed-end conditions on displacements, and it was found that poor correlation was obtained with frequencies given by the theory of the present paper. It is of interest to note the close agreement between inextensional theory and experiment in Saunders' paper on the sphere-cone combination (ref. 10). As shown by Van Urk and Hut (ref. 2) and also observed in the present study, inextensional theory alone is not adequate for the treatment of a completely rigid edge. Apparently, the flexibility introduced at the sphere-cone juncture is sufficient to permit the cone (which represents most of the inertia) to vibrate inextensionally. It is shown herein that this requires that the cone generators remain straight, a condition which cannot be satisfied for a built-in edge.

The present treatment is a generalization of the method of Saunders et al. for the extensional vibrations, using polynomial mode shape expressions which satisfy all fixed-end displacement conditions. An arbitrary number of terms may be included in the polynomials to give any desired degree of accuracy within limits dictated by computational complexity. A simplifying approximation is introduced which permits the extensional and inextensional cases to be computed separately and the results are combined for the general case. Results of this method are compared with experimental results obtained by using induction vibrators.

SYMBOLS

a	radius of fixed end of cone; cylinder radius
A_i	coefficients of assumed mode shape (radial displacement) where $i = 1, 2, 3, \ldots n$
B_i	coefficients of assumed mode shape (tangential displacement) where $i = 1, 2, 3, \ldots n$
C_i	coefficients of assumed mode shape (axial displacement) where $i = 1, 2, 3, \ldots n$
E	Young's modulus of elasticity
f	frequency, cps
h	shell thickness
l	length of cone (or cylinder)
$L = 1 + \lambda \tan \alpha$	
n	number of terms in displacement polynomials
r	radial cylindrical coordinate
s	mode number
t	time
T	kinetic energy
T^*	surface integral associated with kinetic energy, independent of time (see eq. (16))
u	axial displacement component
v	circumferential displacement component
V	strain energy
V^*	surface integral associated with strain energy, independent of time (see eq. (17))
w	radial displacement component
x	axial cylindrical coordinate
α	cone half-angle

Δ combined frequency parameter or eigenvalue (see eq. (22))

Δ_E nondimensional extensional frequency parameter

Δ_I nondimensional inextensional frequency parameter

$\epsilon_1, \epsilon_2, \epsilon_{12}$ middle-surface strains

$\kappa_1, \kappa_2, \kappa_{12}$ principal changes of curvature

$\lambda = \frac{l}{a}$

$\mu = \lambda \tan \alpha$

ν Poisson's ratio

ρ mass density

$\tau = \frac{h}{a}$

ϕ circumferential cylindrical coordinate

ω frequency, rad/sec

Subscripts:

E extensional

I inextensional

Dots over quantities denote differentiation with respect to time.

Primes denote derivative with respect to x.

THEORY OF CONICAL SHELL VIBRATIONS

General Considerations

In studying the vibrations of thin shells, the first question to be considered is whether or not the middle surface of the shell undergoes extension. If extension does not occur, the shell will deform in bending only, and the vibrations are called inextensional or flexural. Vibrations consisting only of stretching deformations are called extensional. In general, both types of vibration will occur simultaneously. However, it can be shown from energy considerations that, for very thin shells, vibration will be predominantly inextensional if the associated deformations are compatible with the prescribed edge conditions of the shell. For example, the strain energy of deformation will be of the form (ref. 12)

$$\text{Strain energy} = Ah\begin{pmatrix}\text{Extensional}\\ \text{deformation}\end{pmatrix}^2 + Bh^3\begin{pmatrix}\text{Bending}\\ \text{deformation}\end{pmatrix}^2 \tag{1}$$

where h is the thickness of the shell and A and B are constants. As h becomes small, the coefficient Bh^3 of the bending term will become much smaller than the coefficient Ah of the stretching term. According to the principle of minimum potential energy, the displacements will be such as to minimize the total strain energy and, therefore, will not involve stretching if the resulting bending deformations are compatible with the boundary conditions.

It will be seen for conical (or cylindrical) shells that only special conditions of edge restraint can satisfy the requirements for inextensional deformations. In particular, these conditions require that the axial generators remain straight. This can occur only if the cone is completely unrestrained or if it is restrained around its circumference in a special manner resembling that of a hinged joint. Any other edge condition will require bending of the axial generators which, as will be shown, cannot occur without some stretching of the middle surface. The fixed edge under investigation here is such a condition and, consequently, both extensional and inextensional deformations must be included.

Inextensional Vibrations

The middle-surface strain-displacement relations for a conical shell, using the nomenclature of figure 1, are given by

$$\left.\begin{aligned}\epsilon_1 &= \frac{\partial u}{\partial x}\cos\alpha\\ \epsilon_2 &= \frac{1}{r}\left(\frac{\partial v}{\partial\phi} - w\cos\alpha + u\sin\alpha\right)\\ \epsilon_{12} &= \frac{\partial v}{\partial x}\cos\alpha - \frac{v\sin\alpha}{r} + \frac{1}{r}\frac{\partial u}{\partial\phi}\end{aligned}\right\} \tag{2}$$

The conditions of inextension of the middle surface require that

$$\epsilon_1 = \epsilon_2 = \epsilon_{12} = 0 \tag{3}$$

which, for free vibrations harmonically varying in time t, are exactly satisfied for the displacement functions

$$\left.\begin{aligned}
u(x,\phi,t) &= -\frac{1}{s}(Aa - B\tan\alpha)\cos\alpha\sin s\phi\sin\omega t \\
v(x,\phi,t) &= (Ax + B)\cos s\phi\sin\omega t \\
w(x,\phi,t) &= -\frac{s}{\cos\alpha}\left[A\left(x + \frac{a\sin\alpha\cos\alpha}{s^2}\right) + B\left(1 - \frac{\sin^2\alpha}{s^2}\right)\right]\sin s\phi\sin\omega t
\end{aligned}\right\} \quad (4)$$

The parameter s is the number of circumferential waves or "mode number." The value $s = 1$ represents a simple rigid body translation which, for inextensional vibrations, involves no deformation of the shell and, therefore, corresponds to zero frequency. The first few even and odd modes are shown in figure 2.

It can be seen from equations (4) that the displacement functions which satisfy the inextensional conditions (eq. (3)) are constant with respect to x for the axial component u and linear in x for the tangential and radial components v and w. This indicates that a conical (or cylindrical) shell can only deform inextensionally if the generators remain straight as, for example, for a freely suspended shell. Any other conditions of edge restraint which prevent motion of the type indicated by equations (4) would require some extension of the middle surface.

For the fixed-end cone of figure 1, all three displacements should vanish at $x = 0$ as well as the slope of w; that is

$$\left.\begin{aligned}
u(0,\phi,t) = v(0,\phi,t) = w(0,\phi,t) &= 0 \\
\left(\frac{\partial w}{\partial x}\right)_{(0,\phi,t)} &= 0
\end{aligned}\right\} \quad (5)$$

It is seen from equations (4) that only one of these conditions may be satisfied. Consequently, the fixed-end cone does not strictly lend itself to inextensional analysis. By selecting, as an approximation, the condition that the radial displacement w vanishes at $x = 0$, equations (4) reduce to

$$\left.\begin{aligned}
u(x,\phi,t) &= \frac{Aa\cos\alpha}{s\left(1 - \frac{\sin^2\alpha}{s^2}\right)}\sin s\phi\sin\omega t \\
v(x,\phi,t) &= -A\left[x - \frac{a\sin\alpha\cos\alpha}{s^2\left(1 - \frac{\sin^2\alpha}{s^2}\right)}\right]\cos s\phi\sin\omega t \\
w(x,\phi,t) &= \frac{Asx}{\cos\alpha}\sin s\phi\sin\omega t
\end{aligned}\right\} \quad (6)$$

where the coefficient A, which represents the amplitude of the motion, is arbitrary. The second term in the brackets of the v displacement is quite small for small cone half-angles (and/or for large values of s). In the limit as the half-angle approaches zero (i.e., for a cylindrical shell), the v displacement also vanishes at $x = 0$ leaving two of the boundary conditions unsatisfied. Nevertheless, it will be shown that beyond a certain value of s the boundary conditions become unimportant and inextensional theory is adequate for predicting the natural frequencies. For lower values of s the extensional frequencies will be shown to predominate.

The inextensional frequencies are obtained from the mode shapes (eqs. (6)) by equating the maximum values of potential and kinetic energies. The potential energy, or strain energy of bending, is expressed in terms of the principal changes of curvature, which are defined by

$$\left.\begin{aligned}
\kappa_1 &= \frac{\partial^2 w}{\partial x^2}\cos^2\alpha \\
\kappa_2 &= \frac{\cos\alpha}{r^2}\frac{\partial v}{\partial \phi} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\sin\alpha\cos\alpha}{r}\frac{\partial w}{\partial x} \\
\kappa_{12} &= \frac{\cos^2\alpha}{r}\frac{\partial v}{\partial x} - \frac{\sin\alpha\cos\alpha}{r^2}v + \frac{\cos\alpha}{r}\frac{\partial^2 w}{\partial x\,\partial \phi} - \frac{\sin\alpha}{r^2}\frac{\partial w}{\partial \phi}
\end{aligned}\right\} \tag{7}$$

The total strain energy involves an integration of the changes of curvature over the surface of the shell according to

$$V_I = \frac{Eh^3}{24(1-\nu^2)}\int_{\phi=0}^{2\pi}\int_{x=0}^{l}\left[\kappa_1{}^2 + \kappa_2{}^2 + 2\nu\kappa_1\kappa_2 + 2(1-\nu)\kappa_{12}{}^2\right]\frac{r\,d\phi\,dx}{\cos\alpha} \tag{8}$$

where E and ν are Young's modulus of elasticity and Poisson's ratio, respectively, and the other parameters are shown in figure 1. Similarly, the kinetic energy is an integral over the surface of the shell of the square of the velocity and is given by

$$T = \frac{1}{2}\rho h\int_{\phi=0}^{2\pi}\int_{x=0}^{l}\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right)\frac{r\,d\phi\,dx}{\cos\alpha} \tag{9}$$

where ρ is the mass density of the shell and the dot denotes differentiation with respect to time. On evaluating these integrals by use of the displacements of equations (6) and equating maximum values of potential and kinetic energies, there is obtained for the inextensional frequencies

$$\omega_I = \frac{hs(s^2-1)}{2a^2\cos\alpha}\left[\frac{E}{3\rho(1-\nu^2)}\right]^{1/2}\left(\frac{N}{D}\right)^{1/2} \tag{10}$$

where

$$N = \left(1 - \frac{\sin^2\alpha}{s^2}\right)^2 \ln L - \frac{2}{L}(L - 1)\left(1 - \frac{\sin^2\alpha}{s^2}\right) + \frac{\left(L^2 - 1\right)}{2L^2} + \frac{(1 - \nu)\left(L^2 - 1\right)\sin^2\alpha}{s^2L^2}$$

$$D = \frac{1}{2}\left(L^2 - 1\right)\left(1 + \frac{s^2}{\cos^2\alpha} + \frac{\tan^2\alpha}{s^2} - 2\tan^2\alpha\right) - \frac{2}{3}\left(L^3 - 1\right)\left(1 - \frac{\sin^2\alpha}{s^2}\right)\left(1 + \frac{s^2}{\cos^2\alpha}\right.$$

$$\left. - \tan^2\alpha\right) + \frac{1}{4}\left(L^4 - 1\right)\left(1 - \frac{\sin^2\alpha}{s^2}\right)^2\left(1 + \frac{s^2}{\cos^2\alpha}\right)$$

$$L = 1 + \lambda \tan \alpha$$

$$\lambda = \frac{l}{a}$$

In the limiting case of zero cone half-angle ($\alpha = 0$), equation (10) reduces to the simple result

$$\omega_I = \frac{hs(s^2 - 1)}{2a^2}\left[\frac{E}{3\rho(1 - \nu^2)}\right]^{1/2}\left[\frac{\dfrac{s^2 + 6(1 - \nu)}{\lambda^2}}{\dfrac{3}{\lambda^2} + s^2(s^2 + 1)}\right]^{1/2} \tag{11}$$

for the inextensional vibrations of a fixed-end, free-end cylinder.

Equation (10) for the conical shell was evaluated over a range of geometries and the results are presented in figures 3(a) to (c) in terms of a nondimensional frequency parameter $\frac{\Delta_I}{\lambda^4\tau^2}$, where $\tau = \frac{h}{a}$ and

$$\Delta_I = \frac{(1 - \nu^2)\rho\lambda^4 a^2 \omega_I^{\,2}}{E} \tag{12}$$

Values of Δ_I from which figures 3(a) to (c) were plotted are given in table I. It can be seen from equation (10) that the inextensional frequencies are directly proportional to the shell thickness and approximately inversely proportional to the square of the radius. This dependence on radius accounts for the large variation in the frequency parameter with cone half-angle in figures 3(a) to (c) since, for a specified fixed-end radius a, the free-end radius, where deflections are maximum, is proportional to the half-angle.

Extensional Vibrations

It was seen in the foregoing analysis that the conditions of inextension of the middle surface (eq. (3)), were sufficient to describe the functional form of the inextensional displacement components. For extensional vibrations, conditions such as these do not exist. Consequently, the extensional mode shapes are not known and must be approximated by assumed modes.

By following the approach of Saunders et al. (ref. 11), the mode shapes are represented by polynomials in the coordinate x, selected to satisfy the fixed-end conditions (eqs. (5)). The free-end conditions are less important since, as will be seen, the maximum extensions occur at the fixed end. The extensional displacement functions, analogous to equations (4) for the inextensional displacements, are then written in the form

$$\left.\begin{aligned} w(x,\phi,t) &= \left(A_1x^2 + A_2x^3 + A_3x^4 + \ldots + A_nx^{n+1}\right)s \sin s\phi \sin \omega t \\ v(x,\phi,t) &= \left(B_1x + B_2x^2 + B_3x^3 + \ldots + B_nx^n\right)\cos s\phi \sin \omega t \\ u(x,\phi,t) &= \left(C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n\right)\frac{\sin s\phi}{s} \sin \omega t \end{aligned}\right\} \tag{13}$$

where the coefficients A_i, B_i, and C_i must be determined from energy considerations. The extensional strain energy is an integral of the extensional strains (eqs. (2)), similar to equation (8) for the bending energy in terms of changes of curvature, and is given by

$$V_E = \frac{Eh}{2(1 - \nu^2)} \int_{\phi=0}^{2\pi} \int_{x=0}^{l} \left[\epsilon_1{}^2 + \epsilon_2{}^2 + 2\nu\epsilon_1\epsilon_2 + \frac{(1 - \nu)}{2} \epsilon_{12}{}^2\right] \frac{r\, d\phi\, dx}{\cos \alpha} \tag{14}$$

It can be shown from the principle of the conservation of energy and use of the calculus of variations that the mode shapes which most nearly approximate the true normal modes are those which render ω^2 (the square of the natural frequency) stationary with respect to the coefficients A_i, B_i, and C_i. This is equivalent to the conditions

$$\left.\begin{aligned} \frac{\partial\omega^2}{\partial A_i} &= 0 \\ \frac{\partial\omega^2}{\partial B_i} &= 0 \\ \frac{\partial\omega^2}{\partial C_i} &= 0 \qquad (i = 1, 2, \ldots n) \end{aligned}\right\} \tag{15}$$

If the mode shapes (eqs. (13)) are substituted into the kinetic energy expression (eq. (9)), the kinetic energy can be written in the form

$$T(x,\phi,t) = \omega^2 T^*(x,\phi)\cos^2\omega t \tag{16}$$

where $T^*(x,\phi)$ is a kinetic energy integral over the surface of the shell and is independent of time. The potential energy (eq. (14)) can be written in a similar form

$$V_E(x,\phi,t) = V^*(x,\phi)\sin^2\omega t \tag{17}$$

where, in like manner, $V^*(x,\phi)$ is a potential energy integral over the surface of the shell, independent of time. Since the maximum values of potential and kinetic energies must be equal, the square of the frequency is found to be

$$\omega^2 = \frac{V^*(x,\phi)}{T^*(x,\phi)} \tag{18}$$

Differentiating ω^2 with respect to A_i gives

$$\frac{\partial\omega^2}{\partial A_i} = \frac{T^* \dfrac{\partial V^*(x,\phi)}{\partial A_i} - V^* \dfrac{\partial T^*(x,\phi)}{\partial A_i}}{T^{*2}} = 0$$

which, with equation (18), may be written

$$\frac{\partial}{\partial A_i}\left[V^*(x,\phi) - \omega^2 T^*(x,\phi)\right] = 0 \tag{19}$$

Expressions with B_i and C_i can be obtained in a similar manner. The operations of equation (19) result in a set of $3n$ linear homogeneous equations in the $3n$ coefficients A_i, B_i, and C_i. For a nontrivial solution, the determinant of the coefficients must vanish; this condition constitutes an eigenvalue problem in ω^2. The characteristic determinant for extensional vibrations then has the following form and is symmetric about the main diagonal:

$$\left|\begin{array}{c|c|c} a_{ij}\,\Delta_E + b_{ij} & c_{ij} & d_{ij} \\ \hline c_{ij} & e_{ij}\,\Delta_E + f_{ij} & g_{ij} \\ \hline d_{ij} & g_{ij} & h_{ij}\,\Delta_E + k_{ij} \end{array}\right| = 0 \tag{20}$$

where i and $j = 1, 2, \ldots n$.

The coefficients of the determinant are given in appendix A. There are nine subsections which may be identified with various types of coupling between the displacements u, v, and w. For the coefficients of appendix A, the coupling between displacement components is as follows:

w - w	w - v	w - u
v - w	v - v	v - u
u - w	u - v	u - u

The coefficients a_{ij}, e_{ij}, and h_{ij} represent contributions to the kinetic energy from the displacements w, v, and u, respectively. Previous investigators have neglected the inertia terms in the axial and circumferential directions, which would correspond here to setting e_{ij} and h_{ij} equal to zero. Breslavskii (see discussion of ref. 13) showed for a cylinder that this is permissible for $s \geqq 3$. It can be seen from a comparison of the magnitudes of the coefficients a_{ij}, e_{ij}, and h_{ij} in appendix A that the coefficients e_{ij} are of the order $1/s^2$ and the coefficients h_{ij} are of the order $1/\lambda^2 s^4$ of the coefficients a_{ij}. Since $\lambda \geqq 2$ for most cones of practical interest, the coefficients h_{ij} may be neglected for all values of s except, possibly, for the rigid body mode $s = 1$, depending on the size of λ. The magnitudes of e_{ij} in comparison with a_{ij} substantiate the observation of Breslavskii that circumferential inertia may be neglected for $s \geqq 3$.

The extensional frequencies were computed by using an IBM 7090 program to evaluate the characteristic determinant (eq. 20)). All inertia terms were retained and numerical data were obtained over a range of cone geometries with six terms in each of the displacement polynomials. The dependence of frequency on number of terms included in the displacement polynomials and limitations of the computer program are discussed in appendix B. Computed extensional frequencies covering the same range of geometries for which the inextensional frequencies were computed are plotted in figures 4(a) to (c) in terms of the parameter Δ_E, which is defined by equation (12) with the subscript E in place of I. Numerical values from which figures 4(a) to (c) were plotted are given in table II. It can be seen from a comparison of these curves for different values of λ that the extensional frequencies are relatively independent of λ. It is also verified from equations (9), (14), and (18) that the frequencies are independent of the shell thickness h. Since Δ_E is proportional to a^2, it is concluded that the only significant geometry effect is an inverse dependence of the extensional frequency on the fixed-end radius a.

Combining of Extensional and Inextensional Frequencies

A rigorous solution of the general vibration problem in which both extensional and inextensional deformations occur would require that the total strain energy contain both the extensional strains and changes of curvature; that is,

$$
\begin{aligned}
V &= V_E + V_I \\
&= \frac{Eh}{2\left(1 - \nu^2\right)} \int_{\phi=0}^{2\pi} \int_{x=0}^{l} \left\{ \left[\epsilon_1^2 + \epsilon_2^2 + 2\nu\epsilon_1\epsilon_2 + \frac{(1 - \nu)}{2} \epsilon_{12}^2\right] \right. \\
&\quad \left. + \frac{h^2}{12}\left[\kappa_1^2 + \kappa_2^2 + 2\nu\kappa_1\kappa_2 + 2(1 - \nu)\kappa_{12}^2\right] \right\} \frac{r\, d\phi\, dx}{\cos \alpha}
\end{aligned}
\tag{21}
$$

The solution would proceed exactly as for the extensional case, except now equation (21) would be used for the strain energy in place of equation (14) for purely extensional deformations. It is seen from a comparison of the extensional strains and changes of curvature (eqs. (2) and (7)) that the bending terms in equation (21) would add considerably to the complexity of the problem. The results of figures 4(a) to (c) show that the purely extensional frequencies are maximum at $s = 1^*$ and decrease rapidly with increasing mode number. It can be shown that the extensional frequency parameter Δ_E is approximately inversely proportional to the third or fourth power of the mode number s. Similarly, figures 3(a) to (c) show that the inextensional frequencies increase monotonically with s, the frequency parameter Δ_I being approximately proportional to the fourth or fifth power of the mode number. Consequently, a plot of actual frequency as a function of mode number will show a minimum frequency at a mode number for which the extensional and inextensional frequencies are approximately equal. Also, since both extensional and inextensional frequencies depend strongly on s, only the two or three frequencies in the vicinity of the minimum frequency will represent comparable magnitudes of both extensional and inextensional deformations if the minimum occurs at a relatively low mode number. The remaining frequencies will be either predominantly extensional or predominantly inextensional depending on whether the frequencies lie to the left or to the right of the minimum, respectively.

In view of the foregoing observations, a simple approximation is proposed for obtaining the frequency response of a fixed-end, free-end conical shell in terms of the extensional and inextensional frequencies, each calculated separately. If it is postulated that the kinetic energy (eq. (9)) is the same for

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*Excluding the "breathing mode," s = 0, which will have a greater frequency.

the extensional and inextensional cases, then the combined frequency parameter Δ is approximately the sum of the extensional and inextensional frequency parameters; that is,

$$\Delta \approx \Delta_E + \Delta_I \tag{22}$$

where Δ_E or Δ_I is defined in accordance with equation (12). (Note that the total strain energy corresponding to equation (22) is simply the sum of the extensional and inextensional components as given in equation (21).) The validity of this approximation stems from the fact that the difference between the extensional and inextensional mode shapes has little effect on the total kinetic energy, for the following reasons:

(1) Extensional strain energy is directly proportional to shell thickness, whereas inextensional or bending energy is proportional to the third power of thickness. Consequently, extremely small extensional displacements, in comparison with bending displacements, will appreciably influence the total strain energy and frequency without causing a significant change in the displacement vector and, hence, the kinetic energy.

(2) Extensional deformations are maximum at the fixed end of the shell where all displacements and, in particular, the radial displacements tend to zero. Since kinetic energy is proportional to the square of the velocity, the greatest contribution to the kinetic energy occurs at the free end of the shell where displacements are maximum and are less influenced by the fixed-end conditions.

Approximations

Beam analogy for cylinder.- If the displacements are represented by the expressions

$$\left.\begin{aligned} u(x,\phi,t) &= \frac{af'(x)}{s} \sin s\phi \sin \omega t \\ v(x,\phi,t) &= -f(x)\cos s\phi \sin \omega t \\ w(x,\phi,t) &= f(x)s \sin s\phi \sin \omega t \end{aligned}\right\} \tag{23}$$

where $f(x)$, the axial mode shape, satisfies the conditions

$$f(0) = f'(0) = 0 \tag{24}$$

then all the fixed-end displacement conditions (eqs. (5)) will be satisfied with the additional restraint (which is not, in general, true) that $v' = 0$ at $x = 0$. For the fixed-end, free-end cylinder obtained by setting $\alpha = 0$ and $r = a$ in the strain-displacement relations (eqs. (2)), the mode shapes of equations (23) will cause the strains ϵ_2 and ϵ_{12} to vanish, leaving ϵ_1 the only nonvanishing component.* The extensional strain energy then reduces to

$$V = \frac{Eha}{2\left(1 - \nu^2\right)} \int_{\phi=0}^{2\pi} \int_{x=0}^{l} \epsilon_1{}^2 d\phi \, dx = \frac{\pi Eha^3 \sin^2\omega t}{2\left(1 - \nu^2\right)s^2} \int_0^l \left[f''(x)\right]^2 dx \tag{25}$$

and the kinetic energy, neglecting the axial component of inertia, becomes

$$T = \frac{1}{2} \rho ha \int_{\phi=0}^{2\pi} \int_{x=0}^{l} \left(\dot{v}^2 + \dot{w}^2\right) d\phi \, dx$$

$$= \frac{1}{2}\pi\rho ha\omega^2\left(s^2 + 1\right)\cos^2\omega t \int_0^l \left[f(x)\right]^2 dx \tag{26}$$

On equating the maximum values of kinetic and potential energies, there is obtained for the purely extensional vibrations

$$\Delta_E = \frac{l^4}{s^2(s^2 + 1)} \frac{\int_0^l \left[f''(x)\right]^2 dx}{\int_0^l \left[f(x)\right]^2 dx} \tag{27}$$

which is precisely the expression for the bending frequencies of a uniform beam of length l, having a ratio of modulus of rigidity to mass per unit length equal to $\frac{l^4}{s^2(s^2 + 1)}$. The ratio of the integrals in equation (27) has a set of values given by $\frac{C_m{}^4}{l^4}$ for the various beam modes, which gives for the extensional vibrations the simple result

$$\Delta_E = \frac{C_m{}^4}{s^2\left(s^2 + 1\right)} \tag{28}$$

Values of C_m, from reference 14, are 0.597π and 1.49π for the first and second cantilever beam modes, respectively.

Inextensional frequencies are found from the previously derived result of equation (11), and the actual frequencies are obtained by combining the extensional and inextensional frequencies by using the simple approximation of equation (22).

A comparison of frequencies calculated by this approximate method with experimental values obtained from reference 15 is given in table III. Also

*In the method presented in reference 11, $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_{12} \neq 0$.

shown, for comparison, are the results of another approximate theory developed by V. I. Weingarten (ref. 15) which also makes use of the vibrating beam results in conjunction with the Donnell differential equation for cylindrical shells. For the first beam mode, results obtained by using the IBM program described previously, with six terms in the displacement polynomials, are also shown for comparison. The frequencies are tabulated for various values of s for the first two beam modes. It is noted that the approximation of combining the extensional and inextensional frequencies gives good results for values of s in the neighborhood of and beyond the s value corresponding to minimum frequency. At lower s values where the extensional frequencies predominate, the results are not as accurate. However, it is significant to note that the disagreement is a result of the inaccuracy of the extensional component and not of the approximation of combining frequencies, which appears in this case to be quite valid.

Cone.- In extending this approximation to the cone, it is no longer possible to make use of the beam analogy because of the additional terms in the strain-displacement relations (eqs. (2)). Nevertheless, the assumptions that the v and w displacements have the same axial functional dependence and that the u displacement is the derivative of this function considerably simplify the selection of mode shapes. For example, if the function $f(x)$ in equations (23) is represented by an nth order polynomial of the form shown for w in equations (13), then the number of arbitrary constants is only one-third of the number required in the general method in which all three displacement polynomials have different coefficients. However, since the mode shapes of equations (23) subject to the conditions of equation (24) will only satisfy the fixed-end conditions for a cylinder, it would be expected that these modes would approximate the fixed-end cone only for small half-angles.

Consider a simple two-term polynomial for $f(x)$ of the form

$$f(x) = Ax^2 + Bx^3 \tag{29}$$

Then,

$$\left.\begin{aligned} u(x,\phi,t) &= \frac{a\left(2Ax + 3Bx^2\right)}{s} \sin s\phi \sin \omega t \\ v(x,\phi,t) &= -\left(Ax^2 + Bx^3\right)\cos s\phi \sin \omega t \\ w(x,\phi,t) &= \left(Ax^2 + Bx^3\right)s \sin s\phi \sin \omega t \end{aligned}\right\} \tag{30}$$

The eigenvalues must satisfy the quadratic equation

$$\Delta_E^2 + \frac{a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12}}{a_{11}a_{22} - a_{12}^2}\Delta_E + \frac{b_{11}b_{22} - b_{12}^2}{a_{11}a_{22} - a_{12}^2} = 0 \tag{31}$$

which has two real, positive roots. The coefficients a_{ij} and b_{ij} corresponding to the mode shapes of equations (30) are given in appendix C.

The approximation involved by taking only two terms for $f(x)$ can be evaluated for the limiting case of zero cone angle by setting α equal to zero and comparing with equation (28) obtained from the cantilever beam analogy for the cylinder. The coefficients of appendix C reduce to

$$\left.\begin{aligned} a_{11} &= \frac{2}{5}\left(s^2 + 1\right) \\ a_{12} &= \frac{1}{3}\left(s^2 + 1\right) \\ a_{22} &= \frac{2}{7}\left(s^2 + 1\right) \end{aligned}\right\} \tag{32a}$$

and

$$\left.\begin{aligned} b_{11} &= -\frac{8}{s^2} \\ b_{12} &= -\frac{12}{s^2} \\ b_{22} &= -\frac{24}{s^2} \end{aligned}\right\} \tag{32b}$$

which give for the smaller root of Δ_E from equation (31) the value

$$\Delta_E = \frac{12.48}{s^2\left(s^2 + 1\right)} \tag{33}$$

The constant $(0.597\pi)^4$ in equation (28) is equal to 12.37, which gives a discrepancy of less than 1 percent in Δ_E.

EXPERIMENTAL STUDIES

Vibration Tests

Vibration frequencies of three small conical shells were experimentally determined by using induction vibrators. Two cones were the same in geometry except for shell thickness. A comparison of the experimentally determined frequencies with the theoretical values predicted by equations (20) and (22) are shown in figures 5(a) to (c), which also include a description of the cone geometries. A comparison of figures 5(a) and 5(b) illustrates the effect of shell thickness on the location of the minimum frequency. Since the geometries are the same, with the exception of shell thickness, the extensional frequencies should be the same, whereas the inextensional frequencies should be 60 percent lower for the 0.006-inch-thick-shell cone; this would shift the minimum frequency to the frequency corresponding to a higher mode number. This is clearly indicated by the experimental results.

A general arrangement of the test setup is shown in figure 6, which includes two induction vibrators mounted adjacent to the 0.010-inch-thick-shell cone that is clamped to a solid table, a stroboscopic lamp for identifying the mode shapes, and a counter for measuring frequencies. The initial test results shown in figure 7 indicate a marked deviation of the lower mode extensional frequencies from the theoretical values. This is believed to result from incomplete end fixity. Subsequent tests, from which the data of figures 5(a) to (c) were obtained, were conducted with the fixed end of the cone imbedded in a large mass of low-melting-point bismuth-tin alloy as shown in figure 8. The experimental results so obtained show acceptable agreement with the theoretically predicted values. The large discrepancy between these two sets of test data, particularly for the $s = 3$ frequency, illustrates that the extensional frequencies are very strongly dependent on the degree of end fixity. For all tests conducted with complete end fixity, neither the rigid body mode $s = 1$ nor the $s = 2$ mode could be detected. This can possibly be explained in terms of a nonlinear effect associated with a localized buckling of the shell, which is described in the next section.

Figures 9 and 10 show two of the circumferential vibration modes obtained by a photographic procedure utilizing the stroboscopic lamp apparatus shown in figure 6. Also shown in figures 9 and 10 are two small metal elements which were attached to the shell just opposite the vibrators to provide a greater source of induction energy in order to obtain displacements sufficiently large for mode identification. The additional mass was shown to have a negligible effect on the motion and frequencies of the shell (the same frequencies were obtained to within 1 cycle out of 100 to 400 cycles per second with and without the metal elements). It can be seen from figures 9 and 10 that the symmetry of the mode shapes is extremely poor compared with the exact symmetry required in a mathematical analysis. This asymmetry is due primarily to the nonuniformity of the shell around the circumference owing to the very small wall thickness and the unavoidable kinks.

Nonlinear Effects

A plot of the relative magnitudes of the three extensional strain components (defined by eqs. (2)) for the fourth mode of a typical cone configuration ($\lambda = 4$, $\alpha = 15^{\circ}$) is presented in figure 11. It is seen that the maximum extensional strain occurs at the fixed end and consists of the axial component ϵ_1. This strain corresponds to an axial stress which varies in tension and compression around the circumference of the shell as shown in figure 12. The magnitude of this stress, for a given radial displacement at the free end, is inversely dependent on the mode number s and is therefore maximum at the lowest mode. This behavior can be seen from the inextensional mode shapes of equation (6); that is, the axial displacement u which would result at the fixed end ($x = 0$) if the shell were unrestrained is prevented from occurring and, thereby, produces the axial stress. By considering the ratio of u at $x = 0$ to w at $x = l$, it is seen that the axial displacement (and corresponding stress) for a given value of w is inversely proportional to s^2. Since this stress occurs in compression as well as in tension, as shown in figure 12, there will be a critical value of stress for a sufficiently large deflection $w_{critical}$ which can cause localized buckling (or at least nonlinear behavior) of the shell. The thinner the shell and the lower the mode number, the lower will be the radial deflection required for this buckling to occur.

CONCLUDING REMARKS

An approximate method has been presented for predicting the vibrations of thin conical shells rigidly fixed at one end and free at the other. It has been demonstrated that, by treating extensional and inextensional vibrations separately, results can be obtained by a relatively simple analytical procedure. Natural frequencies computed by use of this approximation show good agreement with experimental results.

Extensional vibrations were computed by a Rayleigh-Ritz procedure with assumed modes in the form of simple polynomials. The resulting eigenvalue-eigenvector problem was numerically evaluated with an IBM 7090 data processing machine. The computer program developed for this solution requires double precision accuracy and is limited to an 18×18 matrix, corresponding to a maximum of six terms in each of the three displacement mode-shape polynomials. Upon further examination of the matrix, it appears that the computer program can be modified to permit a greater accuracy. Limitations of the present program appear to result from retaining axial and tangential inertia terms which become quite small compared with corresponding radial inertia terms for large values of the circumferential mode number.

It may be noted that in order to facilitate application of the program beyond a specified circumferential mode number, the axial and circumferential inertia terms may be excluded. This modification will permit the present matrix to be reduced to an unsymmetric matrix one-third the size of the original matrix, and the order of the eigenvalue-eigenvector problem will be reduced accordingly.

Since the present treatment of conical shell vibrations is limited to the fixed-free cone, a logical extension of the theory would be the investigation of other boundary conditions such as fixed-pinned, pinned-pinned, and so forth. Such configurations are of particular importance for an understanding of space-vehicle dynamic response, and relatively little work appears to have been done in this area.

Aerojet-General Corporation,
Azusa, California, January 26, 1965.

APPENDIX A

DETERMINANT ELEMENTS OF EQUATION (20)

The coefficients of the characteristic determinant for extensional vibrations as given in equation (20) are as follows:

$$a_{ij} = 2s^2\left(\frac{1}{3 + i + j} + \frac{\mu}{4 + i + j}\right)$$

$$b_{ij} = A_2 M_{2+i+j}$$

$$c_{ij} = A_1 M_{1+i+j}$$

$$d_{ij} = B_2\left(\mu M_{1+i+j} + \frac{j\nu}{1 + i + j}\right)$$

$$e_{ij} = 2\left(\frac{1}{1 + i + j} + \frac{\mu}{2 + i + j}\right)$$

$$f_{ij} = AM_{i+j} + C_2\left[\frac{ij}{i + j - 1} + \left(\frac{ij}{i + j} - 1\right)\mu + \mu^2 M_{i+j}\right]$$

$$g_{ij} = B_1\left(\mu M_{i+j} + \frac{j\nu}{i + j}\right) + C_1\left(\frac{i}{i + j} - \mu M_{i+j}\right)$$

$$h_{ij} = \frac{2}{(\lambda s)^2}\left(\frac{1}{1 + i + j} + \frac{\mu}{i + j + 2}\right)$$

$$k_{ij} = CM_{i+j} + D\left(\frac{ij}{i + j - 1} + \frac{ij\mu}{i + j} + \mu^2 M_{i+j} + \mu\nu\right)$$

$A = -2\lambda^4 s^2$ $\qquad B_1 = 2\lambda^2 \cos\alpha$ $\qquad C_1 = C\cos\alpha$

$A_1 = A\cos\alpha$ $\qquad B_2 = B_1\cos\alpha$ $\qquad C_2 = C_1\cos\alpha$

$A_2 = A_1\cos\alpha$ $\qquad C = -\lambda^2(1 - \nu)$ $\qquad D = -\dfrac{2\cos^2\alpha}{s^2}$

APPENDIX A

$$M_n = \int_0^1 \frac{x^n}{1 + \mu x}\, dx$$

$$= \frac{(-1)^n}{\mu^{n+1}}\left[\ln (1 + \mu) - \mu + \frac{\mu^2}{2} - \frac{\mu^3}{3} + \ . \ . \ . + (-1)^n \frac{\mu^n}{n}\right]$$

$$= \frac{1}{n + 1} - \frac{\mu}{n + 2} + \frac{\mu^2}{n + 3} - \frac{\mu^3}{n + 4} + \ . \ . \ .$$

APPENDIX B

CHARACTERISTICS OF IBM 7090 PROGRAM

The analytical method and the IBM 7090 computer program for the solution of the characteristic determinant (eq. (20)) were developed under NASA Contract No. NASr-111 by the Aerojet-General Corporation, Space Propulsion Division of the Liquid Rocket Plant, Azusa, California.* The following evaluation is based on the actual performance of the program in computing natural vibration frequencies over a wide range of cone geometries with different numbers of terms included in the displacement polynomials.

The IBM program is used to compute the extensional frequency parameter Δ_E and corresponding mode shapes for specified input values of the following parameters:

s	number of circumferential waves
α	cone half-angle, degrees
λ	ratio of cone length to fixed-end radius
n	number of terms in displacement polynomials

For convenience, the inextensional frequency parameter $\Delta_I/\lambda^4\tau^2$ is also computed and included in the output. A typical print out is shown in table IV. The mode shape parameters A_i, B_i, and C_i are coefficients of the displacement polynomials expressed in terms of a nondimensional axial coordinate x/l and normalized with respect to A_1; that is,

$$w(x,\phi,t) = \left[A_1\left(\frac{x}{l}\right)^2 + A_2\left(\frac{x}{l}\right)^3 + \dots + A_n\left(\frac{x}{l}\right)^{n+1}\right] s \sin s\phi \sin \omega t$$

$$v(x,\phi,t) = \left[B_1\left(\frac{x}{l}\right) + B_2\left(\frac{x}{l}\right)^2 + \dots + B_n\left(\frac{x}{l}\right)^{n}\right] \cos s\phi \sin \omega t$$

$$u(x,\phi,t) = \left[C_1\left(\frac{x}{l}\right) + C_2\left(\frac{x}{l}\right)^2 + \dots + C_n\left(\frac{x}{l}\right)^{n}\right] \frac{\sin s\phi}{\lambda s} \sin \omega t$$

*All changes in the final report submitted by Aerojet-General Corporation under Contract NASr-111(AGC Rept. No. 2581a) are the responsibility of Aerospace Research Associates, Inc., and not Aerojet-General Corporation.

For example, at the free end, $x = l$, the relative maximum values of the three displacement components are

$$w_{max}(l) = \left(A_1 + A_2 + \ldots + A_n\right)s$$

$$v_{max}(l) = \left(B_1 + B_2 + \ldots + B_n\right)$$

$$u_{max}(l) = \frac{\left(C_1 + C_2 + \ldots + C_n\right)}{\lambda s}$$

The only basis for ascertaining the accuracy of a computed frequency (at least within the limitations of the analytical method) is the apparent degree of convergence for a specified number of terms in the displacement polynomials. Since the number of terms is limited to six, the degree of convergence can only be estimated by comparing the results for six terms with those for fewer numbers of terms.

Table V is a summary of frequency results over a wide range of cone geometries, in terms of the nondimensional extensional frequency parameter Δ_E (which is proportional to the square of the natural frequency). Frequencies were computed for the following parameters:

$$s = 1,\ 12$$

$$\alpha = 0^\circ,\ 20^\circ,\ 40^\circ$$

$$\lambda = 1,\ 4,\ 10$$

$$n = 3,\ 4,\ 5,\ 6$$

It is noted that, with the exception of the 40° half-angle cones at $s = 12$, the frequencies appear to be converging toward definite values. For the cases corresponding to $\alpha = 40^\circ$ and $s = 12$, where no values are shown, the computed frequencies were obviously in error, and apparently the program failed. At $s = 1$, the results appear to be valid even for $\alpha = 40^\circ$, but the shorter the cone the better the convergence. It should also be noted that the values of table V are proportional to the square of the frequency and, therefore, the corresponding deviations in frequency are proportionately less.

Limitations of the program appear to result from retaining axial and tangential inertia terms which become quite small compared with corresponding radial inertia terms for large values of s. For example, it can be seen from appendix A that the difference in magnitudes of the coefficients a_{ij}, e_{ij}, and h_{ij}, which represent inertias in the radial, tangential, and axial directions, respectively, becomes progressively greater as s increases. The ratios e_{ij}/a_{ij} and h_{ij}/a_{ij} are proportional to $1/s^2$ and $1/\lambda^2 s^4$, respectively. Consequently, for relatively large values of λ and s, the tangential and axial inertias become unimportant with respect to the radial inertia and apparently give rise to convergence problems. If the coefficients e_{ij} and h_{ij} are assumed to be zero, the eigenvalue Δ_E will not appear in the center and lower diagonal subsections of the characteristic determinant (eq. (20)). The determinant can then be reduced to an unsymmetric determinant one-third the size of the original determinant, and the order of the eigenvalue-eigenvector problem will be reduced accordingly.

APPENDIX C

COEFFICIENTS OF EQUATION (31)

The coefficients of equation (31) for approximate extensional solution of conical shell with two-term polynomial mode shape are as follows:

$$a_{11} = \left(s^2 + 1\right)\left(\frac{2}{5} + \frac{1}{3}\mu\right)$$

$$a_{12} = \left(s^2 + 1\right)\left(\frac{1}{3} + \frac{2}{7}\mu\right)$$

$$a_{22} = \left(s^2 + 1\right)\left(\frac{2}{7} + \frac{1}{4}\mu\right)$$

$$\begin{aligned} b_{11} = &- \frac{\cos\alpha}{s^2}(8 + 4\mu) - 2M_4\lambda^4 s^2(1 - \cos\alpha)^2 - 8M_3\lambda^2\mu\cos\alpha(1 - \cos\alpha) \\ &- \frac{8M_2\mu^2\cos^2\alpha}{s^2} - \frac{2\lambda^2\cos\alpha}{s}\left[\frac{2}{3}s(1 - \cos\alpha) + \frac{2\mu\cos\alpha}{\lambda^2 s}\right] \\ &- 2M_4\lambda^2\mu^2\cos^2\alpha - 8M_3\lambda^2\mu\cos\alpha - 8M_2\lambda^2 + \frac{16}{3}\lambda^2\cos\alpha - \frac{8}{3}\lambda^2\cos^2\alpha \end{aligned}$$

$$\begin{aligned} b_{12} = &- \frac{\cos\alpha}{s^2}(12 + 8\mu) - 2M_5\lambda^4 s^2(1 - \cos\alpha)^2 - 10M_4\lambda^2\mu\cos\alpha(1 - \cos\alpha) \\ &- \frac{12M_3\mu^2\cos^2\alpha}{s^2} - \frac{2\lambda^2\cos\alpha}{s}\left[s(1 - \cos\alpha) + \frac{3\mu\cos\alpha}{\lambda^2 s}\right] - 2M_5\lambda^2\mu^2\cos^2\alpha \\ &- 10M_4\lambda^2\mu\cos\alpha - 12M_3\lambda^2 + 2\lambda^2\mu\cos^2\alpha + 6\lambda^2\cos\alpha - 3\lambda^2\cos^2\alpha \\ &- \frac{12}{5}\lambda^2\mu\cos^2\alpha \end{aligned}$$

APPENDIX C

$$b_{22} = -\frac{\cos\alpha}{s^2}(24 + 18\mu) - 2M_6\lambda^4 s^2(1 - \cos\alpha)^2 - 12M_5\lambda^2\mu\cos\alpha(1 - \cos\alpha)$$

$$-\frac{18M_4\mu^2\cos^2\alpha}{s^2} - \frac{2\lambda^2\cos\alpha}{s}\left[\frac{6}{5}s(1 - \cos\alpha) + \frac{9\mu\cos\alpha}{2\lambda^2 s}\right] - 12M_5\lambda^2\mu\cos\alpha$$

$$- 2M_6\lambda^2\mu^2\cos^2\alpha - 18M_4\lambda^2 + 2\lambda^2\mu\cos^2\alpha + \frac{36}{5}\lambda^2\cos\alpha$$

$$- \frac{18}{5}\lambda^2\cos^2\alpha - 3\lambda^2\mu\cos^2\alpha$$

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TABLE I.- CALCULATED VALUES OF THE NONDIMENSIONAL

INEXTENSIONAL FREQUENCY PARAMETER $\Delta_I/\lambda^4\tau^2$

s	$\Delta_I/\lambda^4\tau^2$ for -		
	$\lambda = 2$	$\lambda = 4$	$\lambda = 6$
		$\alpha = 0^\circ$	
2	0.73012	0.63344	0.61494
3	5.3157	4.9297	4.8577
4	18.753	17.924	17.770
5	48.037	46.625	46.363
6	102.16	100.03	99.640
7	192.13	189.15	188.60
8	330.94	326.98	326.25
9	533.60	528.52	527.58
10	817.09	810.77	809.60
11	1200.4	1192.7	1191.3
12	1704.6	1695.4	1693.7
		$\alpha = 15^\circ$	
2	0.20284	0.067545	0.031107
3	1.4390	.51808	.24305
4	5.0514	1.8770	.88639
5	12.921	4.8759	2.3097
6	27.467	10.455	4.9605
7	51.645	19.761	9.3859
8	88.950	34.153	16.232
9	143.41	55.196	26.245
10	219.60	84.664	40.270
11	322.62	124.54	59.251
12	458.12	177.02	84.233
		$\alpha = 30^\circ$	
2	0.072660	0.015330	0.0054584
3	.49615	.11457	.041697
4	1.7317	.41234	.15105
5	4.4243	1.0683	.39242
6	9.4028	2.2875	.84147
7	17.680	4.3204	1.5907
8	30.452	7.4631	2.7492
9	49.101	12.057	4.44330
10	75.191	18.490	6.8157
11	110.47	27.194	10.026
12	156.88	38.648	14.251

TABLE II.- CALCULATED VALUES OF THE NONDIMENSIONAL

EXTENSIONAL FREQUENCY PARAMETER Δ_E

s	Δ_E for -		
	$\lambda = 2$	$\lambda = 4$	$\lambda = 6$
		$\alpha = 0^\circ$	
1	1.2185	3.025	4.1121
2	.30554	.47569	.52576
3	.093183	.11711	.12218
4	.035172	.040103	.040987
5	.015689	.017029	.017250
6	.0079261	.0083715	.0084422
7	.0043982	.0045703	.0045976
8	.0026237	.0026986	.0027106
9	.0016572	.0016931	.0016989
10	.0010962	.0011147	.0011179
11	.00075315	.00076334	.00076511
12	.00053408	.00054001	.00054107
		$\alpha = 15^\circ$	
1	0.64928	1.4010	1.8720
2	.17989	.27778	.31187
3	.059842	.075373	.078334
4	.023823	.026877	.026998
5	.010963	.011630	.011492
6	.0056430	.0057730	.0056560
7	.0031681	.0031691	.0030898
8	.0019044	.0018776	.0018250
9	.0012091	.0011806	.0011453
10	.00080268	.00077847	.00075421
11	.00055294	.00053366	.00051649
12	.00039283	.00037785	.00036543
		$\alpha = 30^\circ$	
1	0.28515	0.55073	0.72017
2	.079415	.12016	.13922
3	.027255	.034610	.037107
4	.011120	.012685	.013063
5	.0051982	.0055594	.0056078
6	.0027018	.0027771	.0027704
7	.0015260	.0015296	.0015163
8	.00092085	.00090791	.00089657
9	.00058613	.00057149	.00056295
10	.00038977	.00037709	.00037084
11	.00026881	.00025860	.00025406
12	.00019114	.00018316	.00017977

TABLE III.- COMPARISON OF CALCULATED AND EXPERIMENTAL FREQUENCIES

FOR FIXED-FREE CYLINDRICAL SHELLS

[1020 steel; h = 0.010 in.; a = 4 in.; λ = 2.232]

Mode number, s	Inextensional frequency from eq. (11), f_I, cps	Extensional frequency from eq. (28), f_E, cps	Combined frequency, $(f_I{}^2 + f_E{}^2)^{1/2}$, cps	Experimental frequency from ref. 15, cps	Frequency from theory of ref. 15, cps	Frequency obtained by eqs. (20) and (22), cps
First beam mode						
3	46.3	603	604	400	422	513
5	140	224	265	239	219	249
7	280	116	303	304	310	300
8	368	88.7	379	376	396	377
10	579	56.9	581	595	610	582
11	701	47.1	703	713	737	702
12	836	39.6	837	844	876	837
13	982	33.8	983	992	1027	
Second beam mode						
7	280	723	775	837	742	
8	368	553	664	693	666	
9	467	437	640	642	665	
12	836	247	872	855	931	
13	982	211	1004	992	1071	

TABLE IV.- TYPICAL COMPUTER PROGRAM PRINT OUT

Number of Terms = 6
Lambda = 4.00
Alpha = 15.00

DELTA ROOT	RESIDUAL	EIGEN-DELTA (Δ_E)
0.99999999E-02	0.82407334E-07	
0.59999999E-01	0.71415751E-07	
0.11000000E-00	0.44998153E-07	
0.16000000E-00	0.38417154E-08	
0.20999999E-00	-0.50972322E-07	
0.16049999E-00	0.33590571E-08	
0.16099999E-00	0.28750290E-08	
0.16149999E-00	0.23896348E-08	
0.16199999E-00	0.19028754E-08	
0.16249999E-00	0.14147523E-08	
0.16299999E-00	0.92526665E-09	
0.16349999E-00	0.43441983E-09	
0.16399999E-00	-0.57786929E-10	
		0.26876745E-01

Mode Shape Coefficients:

$A_1, A_2, \ldots A_n$
$B_1, B_2, \ldots B_n$
$C_1, C_2, \ldots C_n$

0.09999999E 01	-0.28254224E 01	0.56667109E 01	-0.68597585E 01	0.44116732E 01	-0.11561308E 01
0.94341283E-02	-0.86286796E 00	0.17059352E 01	-0.21329884E 01	0.14602529E 01	-0.40763597E-00
0.10419840E 01	-0.57337014E 00	-0.31412425E 01	0.74117771E 01	-0.65043095E 01	0.20730519E 01

GENERALIZED MASS = 0.34299628E 03

INEXTENSIONAL COMPONENT = 0.18769860E 01 $\left(\Delta_I / \lambda^4 \tau^2\right)$

TABLE V.- DEPENDENCE OF FREQUENCY ON NUMBER OF TERMS IN DISPLACEMENT POLYNOMIALS FOR A WIDE RANGE OF CONE GEOMETRIES

n	s	Δ_E for -		
		$\alpha = 0^\circ$	$\alpha = 20^\circ$	$\alpha = 40^\circ$
		$\lambda = 1$		
3	1	0.31007	0.16353	0.054484
4	1	.30740	.16202	.053756
5	1	.30632	.16138	.053439
6	1	.30576	.16105	.053361
3	12	.00052357	.00034466	.00010327
4	12	.00051130	.00032524	----------
5	12	.00050854	.00032150	----------
6	12	.00050587	.00031909	----------
		$\lambda = 4$		
3	1	3.2163	1.2158	0.31436
4	1	3.1004	1.1118	.27314
5	1	3.0518	1.0721	.25627
6	1	3.0245	1.0533	.24824
3	12	.00054981	.00042593	.00019230
4	12	.00054271	.00032717	.00010826
5	12	.00054093	.00031385	----------
6	12	.00054001	.00031129	----------
		$\lambda = 10$		
3	1	5.2343	2.9482	0.92050
4	1	5.1006	2.1900	.61074
5	1	5.0611	1.9279	.48824
6	1	5.0275	1.8105	.43014
3	12	.00055106	.00074946	.00056632
4	12	.00054483	.00039215	.00021094
5	12	.00054261	.00031483	.00012701
6	12	.00054162	.00029396	----------

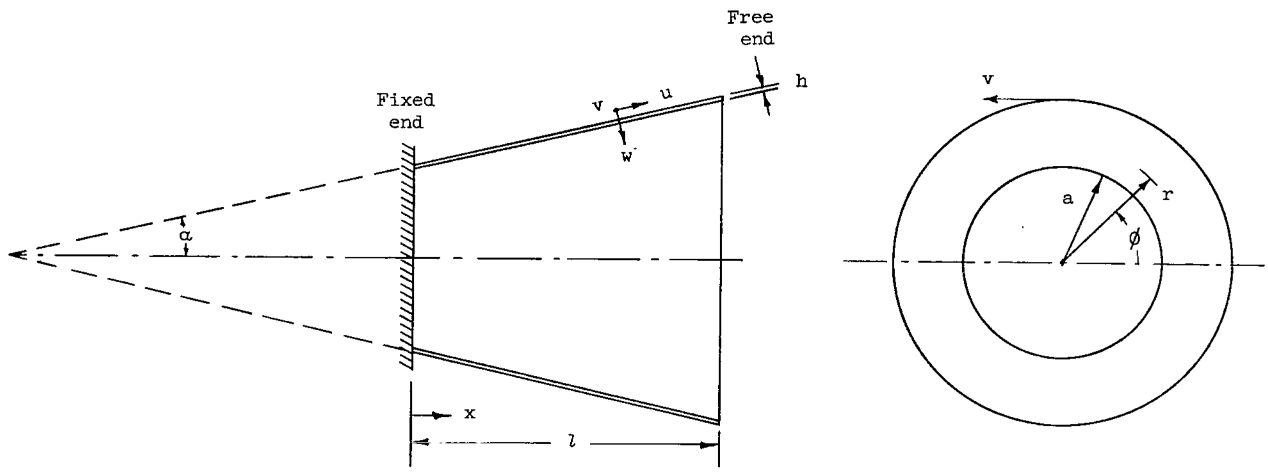


Figure 1.- Conical shell nomenclature.

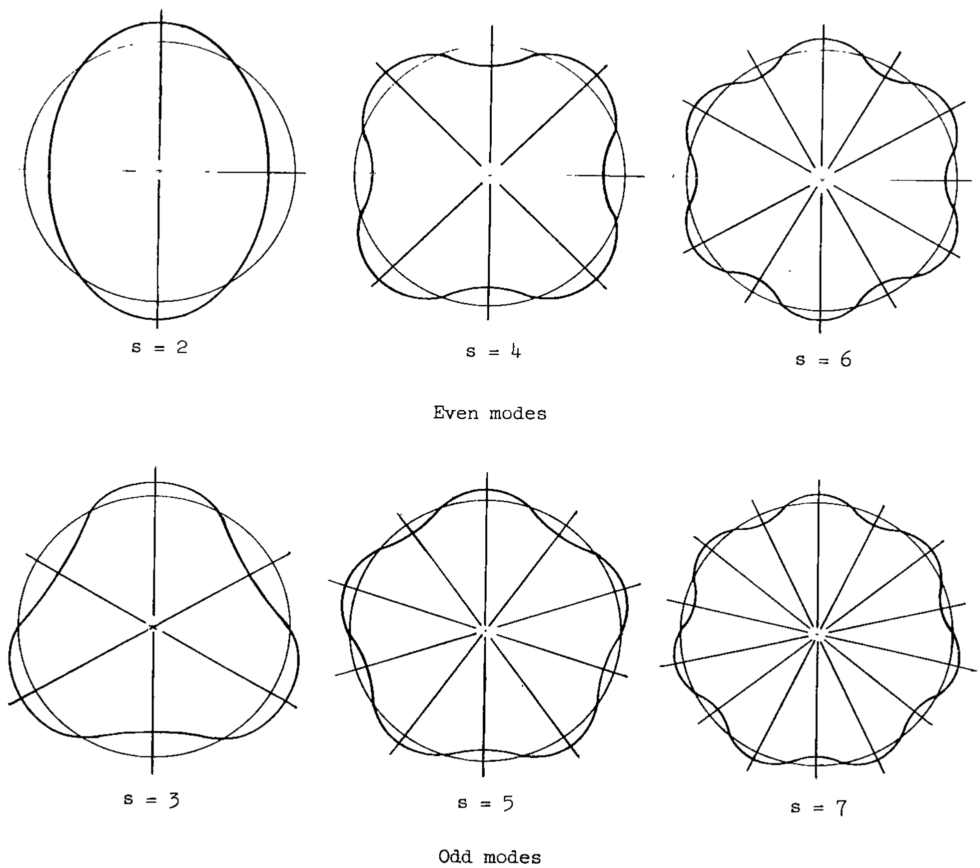
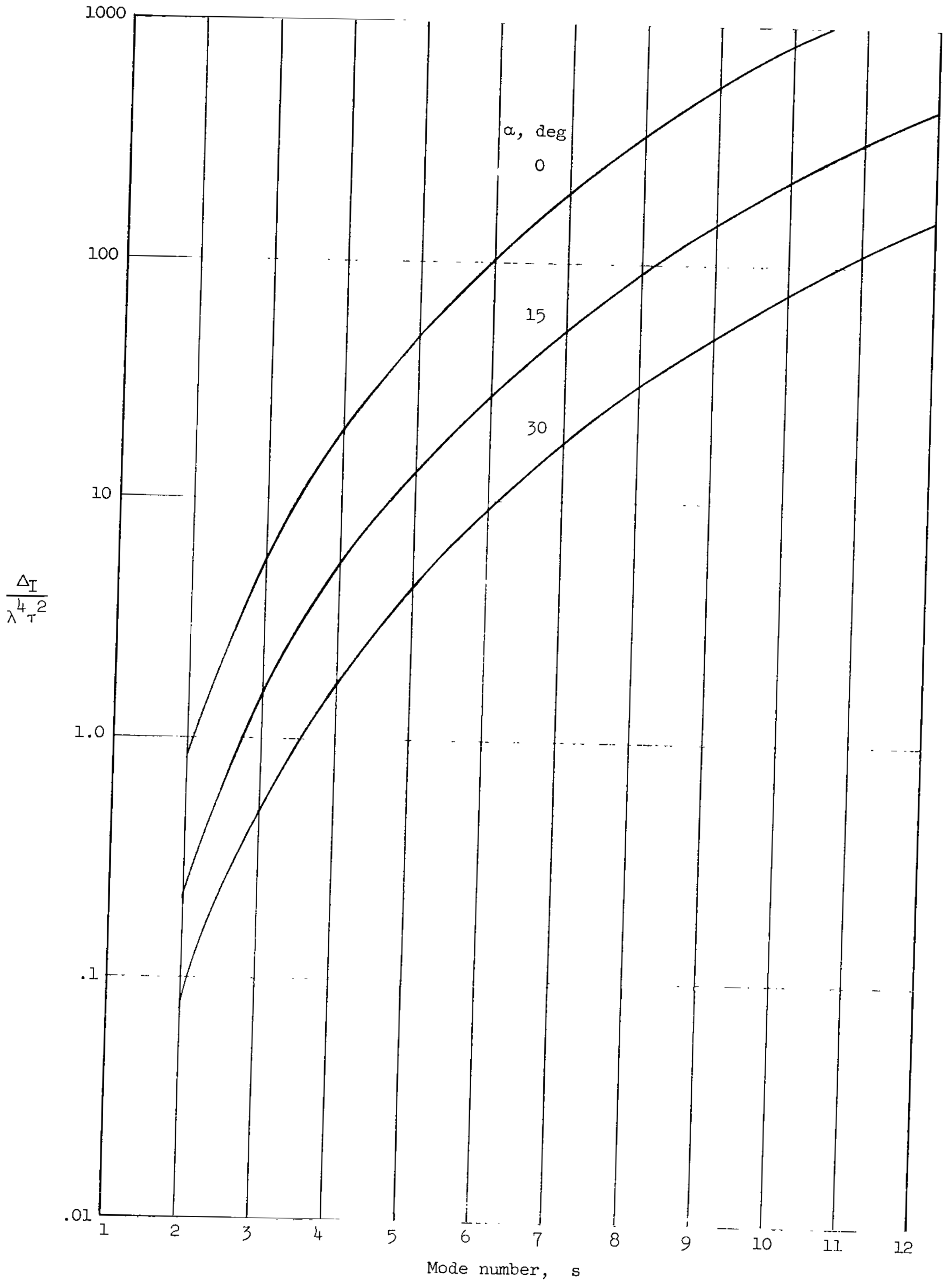
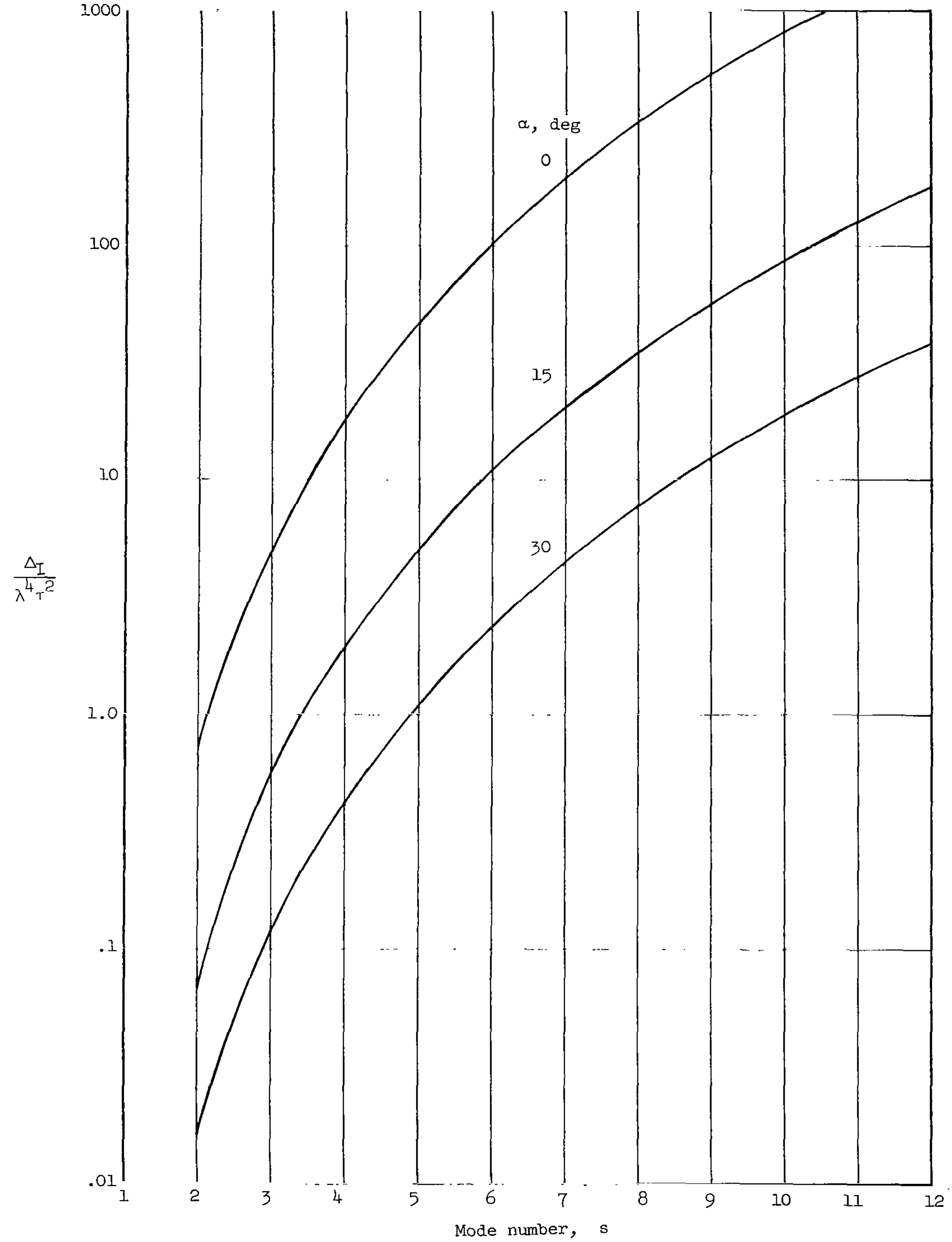


Figure 2.- Circumferential mode shapes.



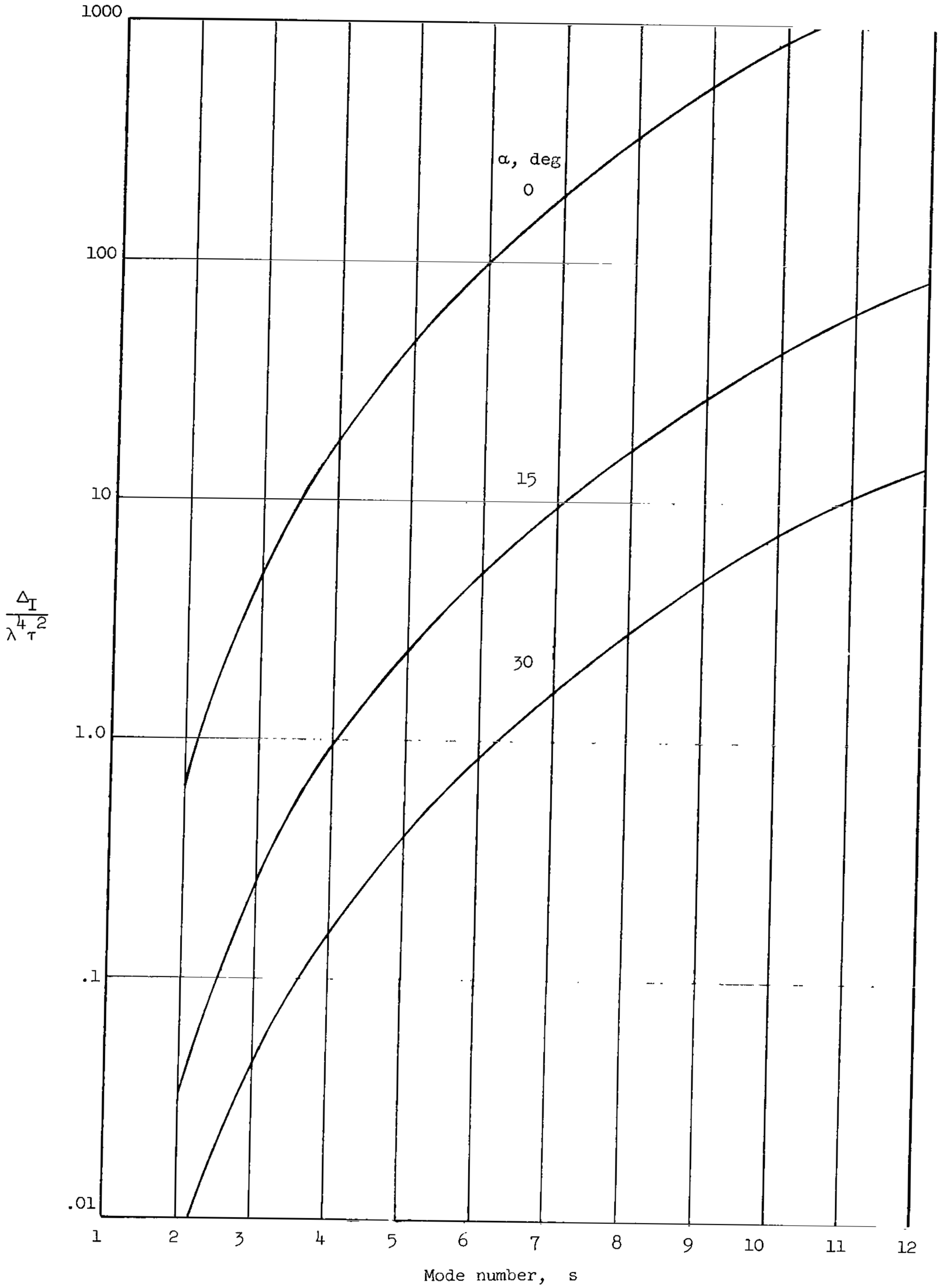
(a) $\lambda = 2$.

Figure 3.- Variation of nondimensional inextensional frequency parameter with mode number.



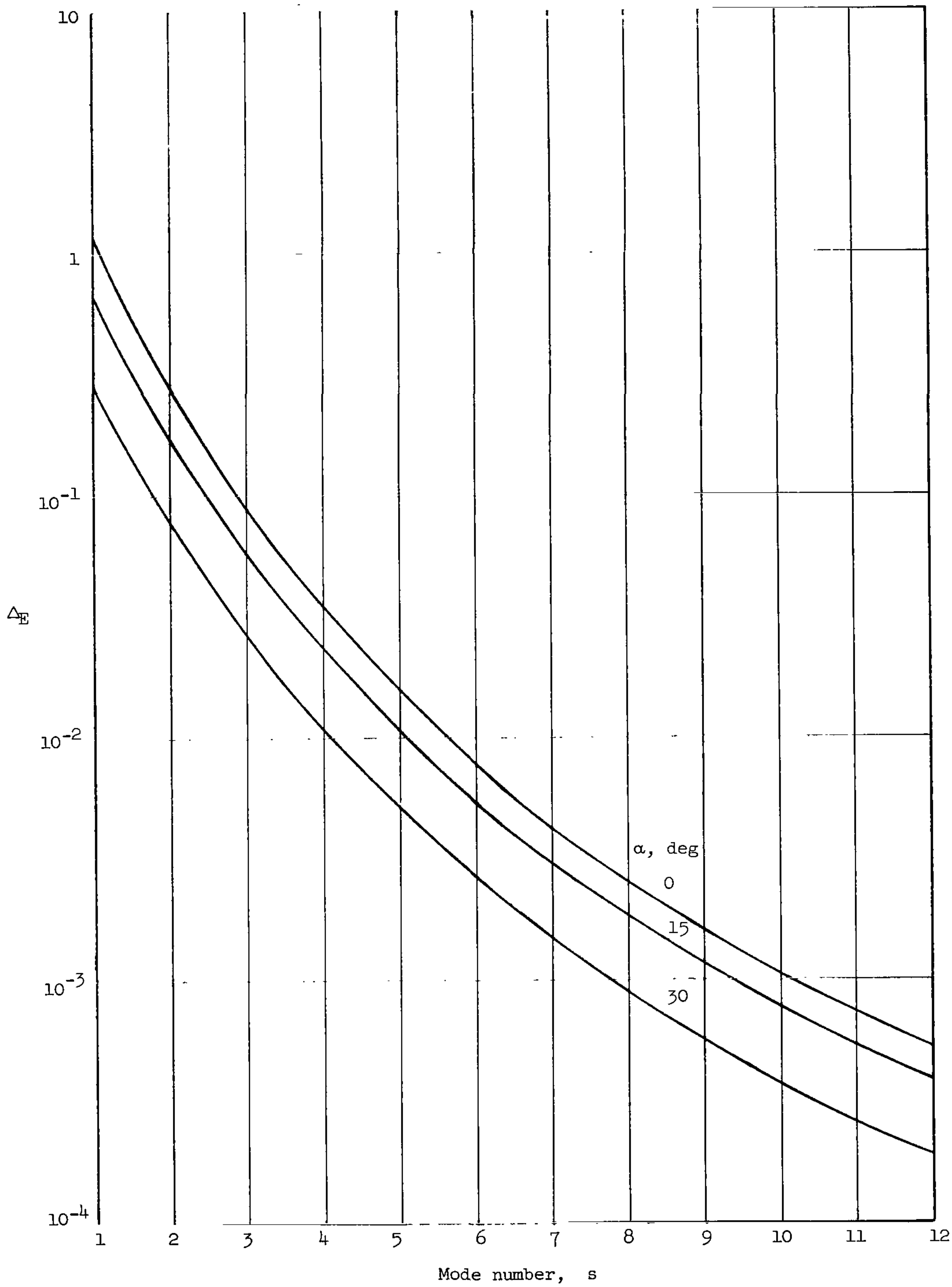
(b) $\lambda = 4$.

Figure 3.- Continued.



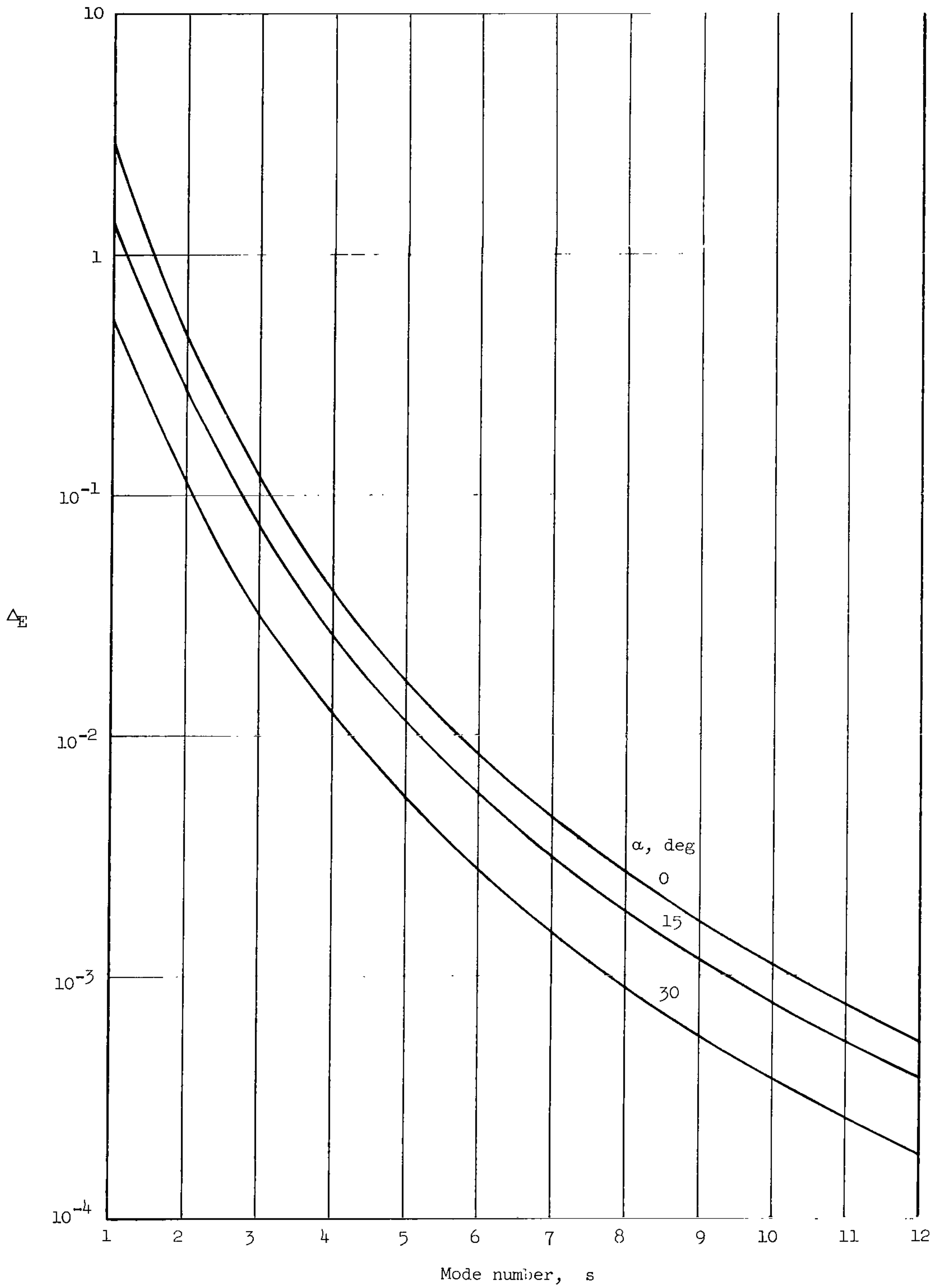
(c) $\lambda = 6$.

Figure 3.- Concluded.



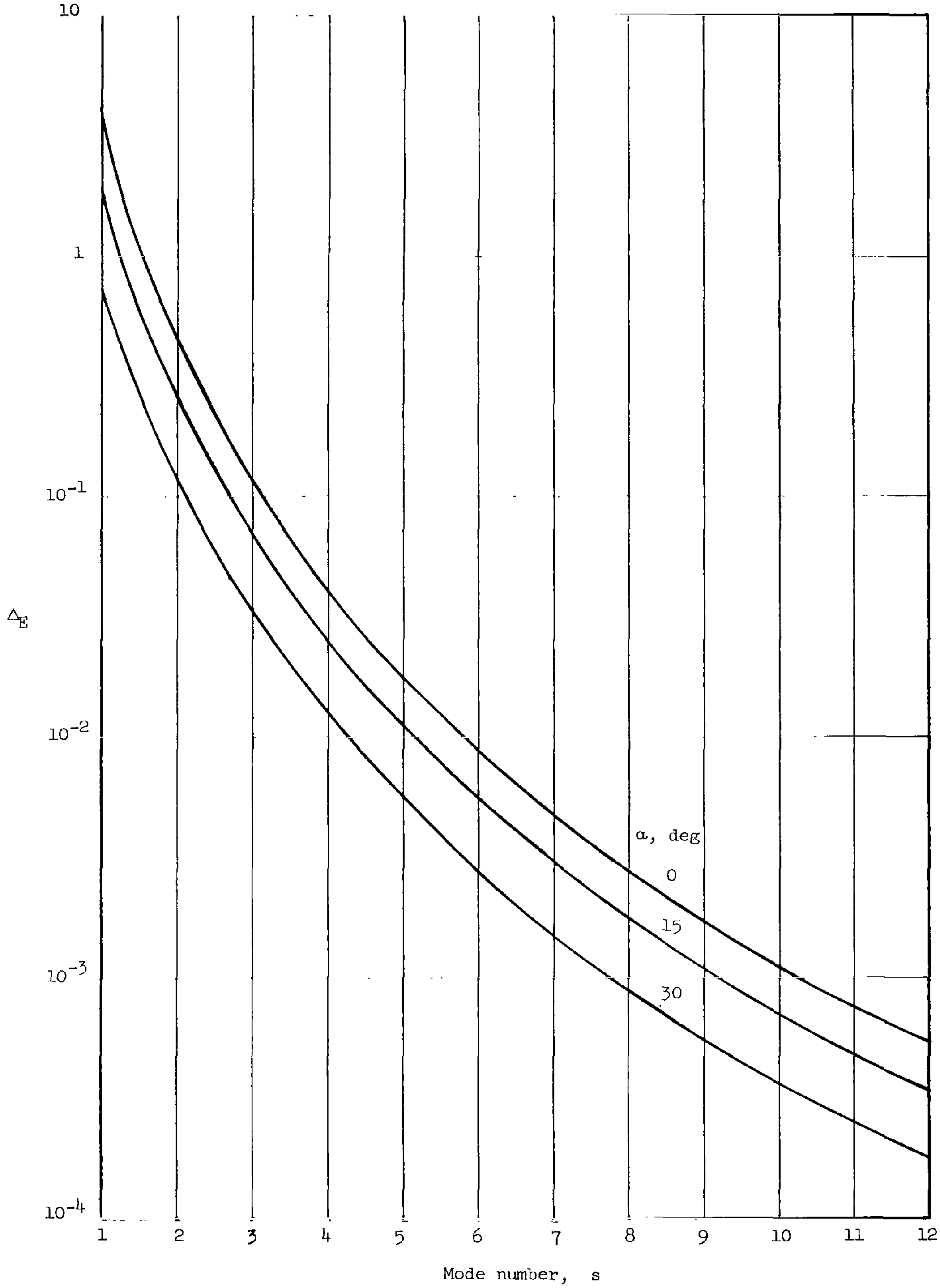
(a) $\lambda = 2$.

Figure 4.- Variation of nondimensional extensional frequency parameter with mode number.



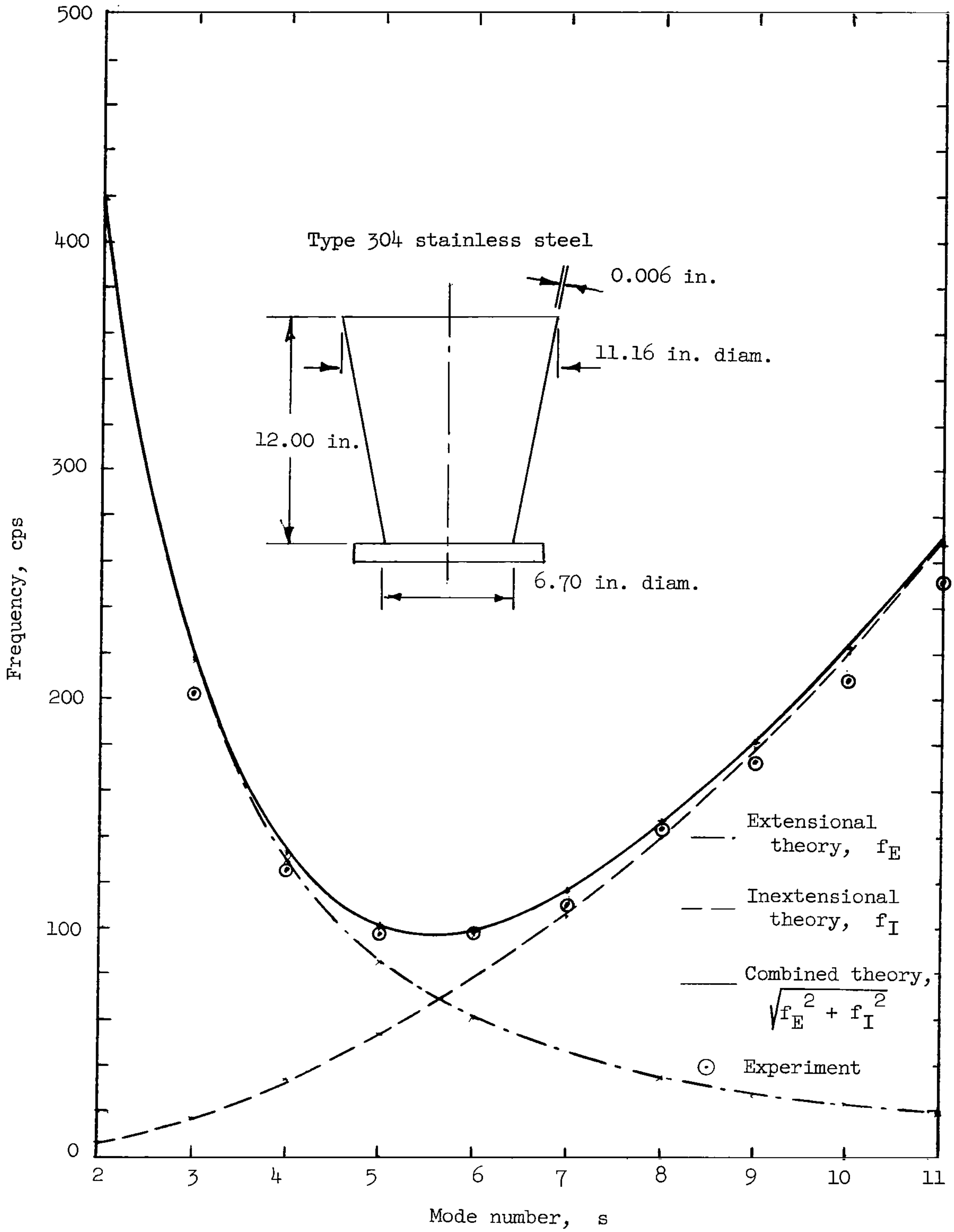
(b) $\lambda = 4$.

Figure 4.- Continued.



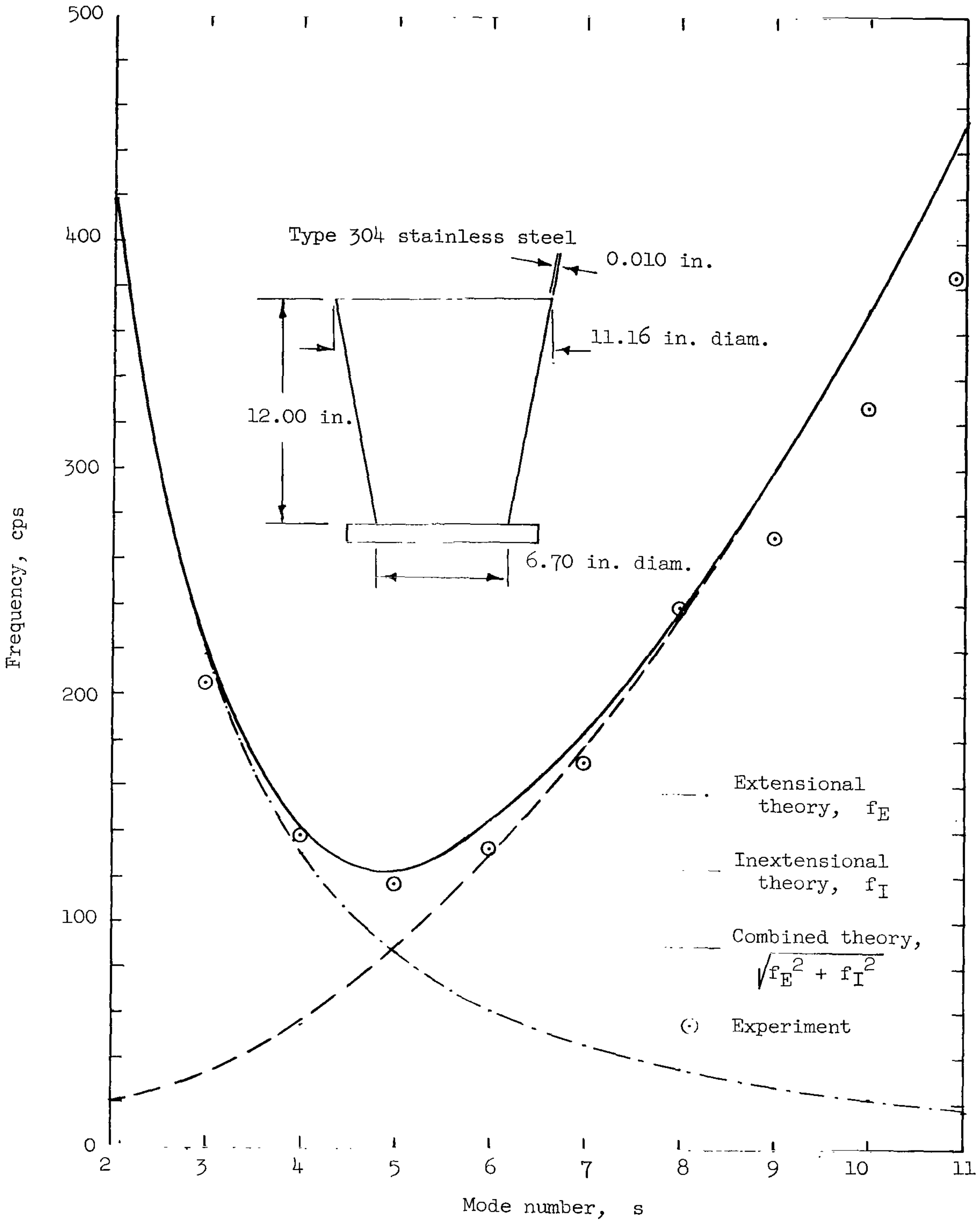
(c) $\lambda = 6$.

Figure 4.- Concluded.



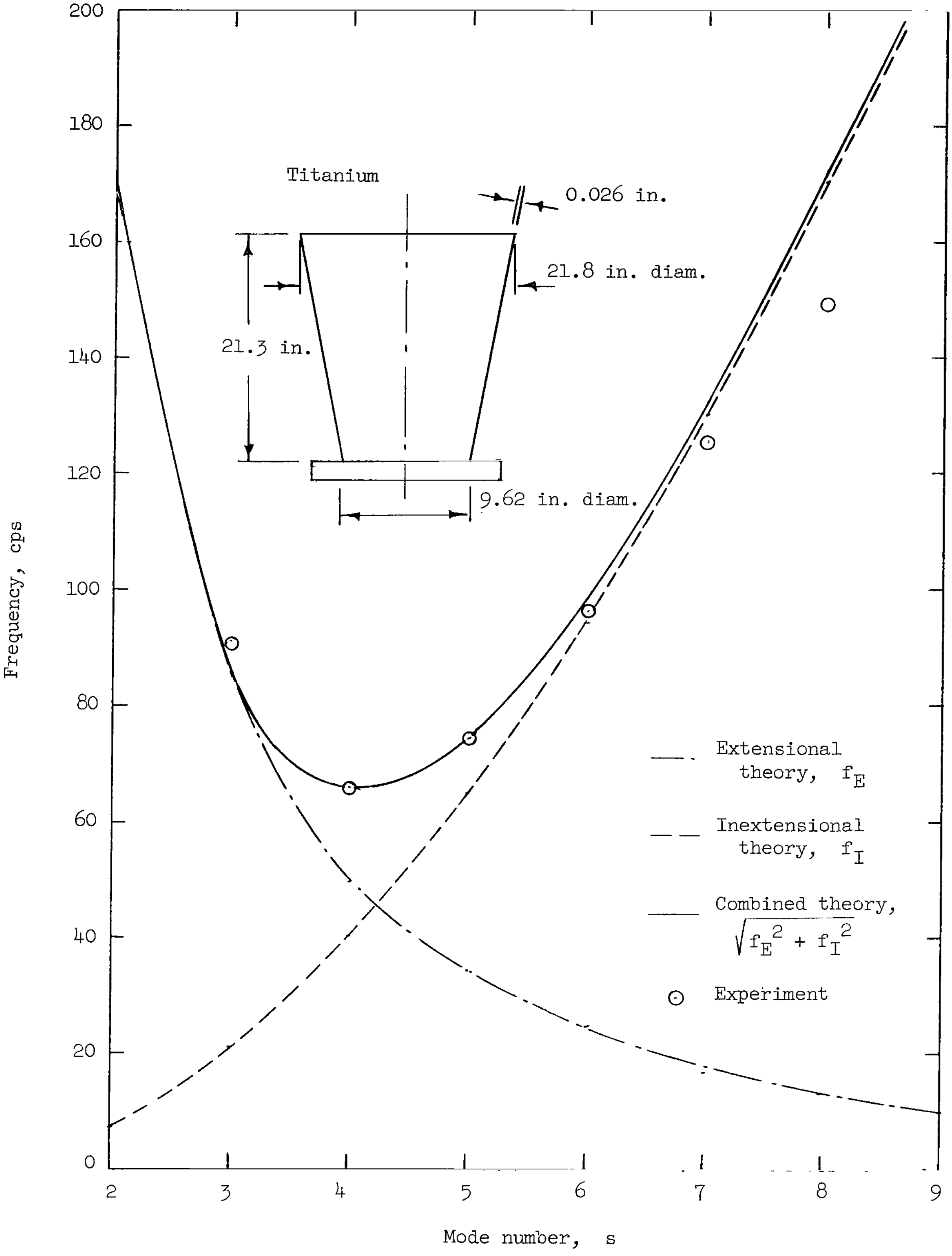
(a) h = 0.006 in.

Figure 5.- Experimental and theoretical frequencies of type 304 stainless-steel cone.



(b) h = 0.010 in.

Figure 5.- Continued.



(c) h = 0.026 in.

Figure 5.- Concluded.

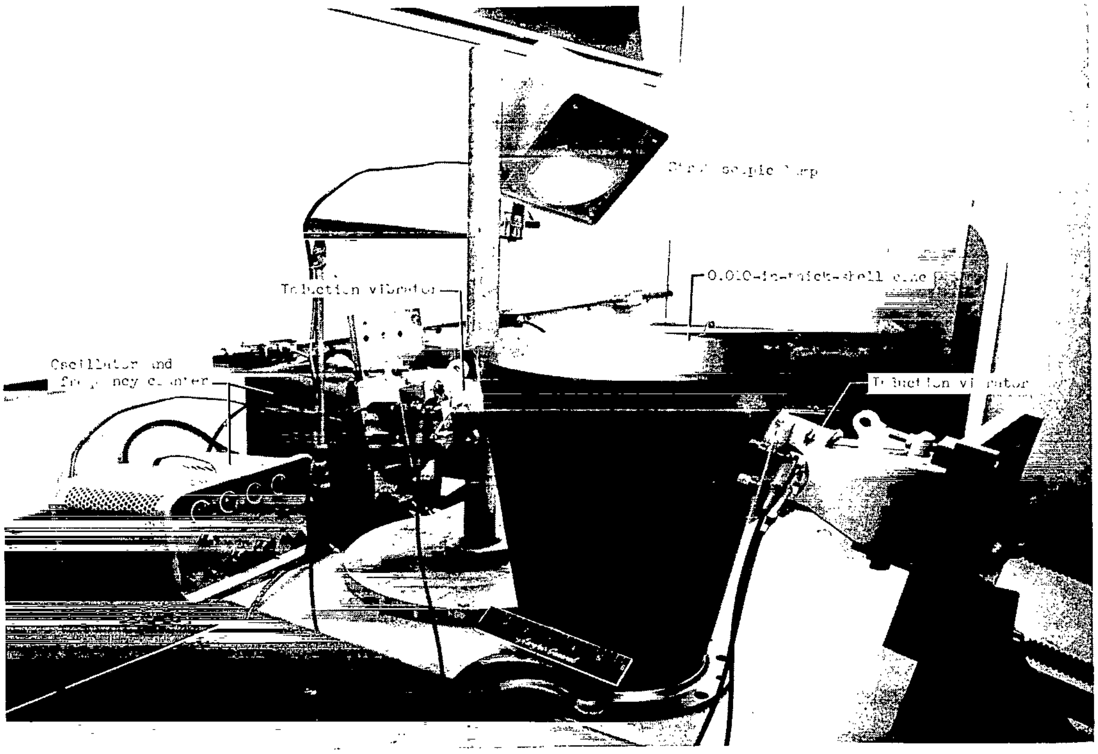


Figure 6.- General test instrumentation arrangement. L-65-23

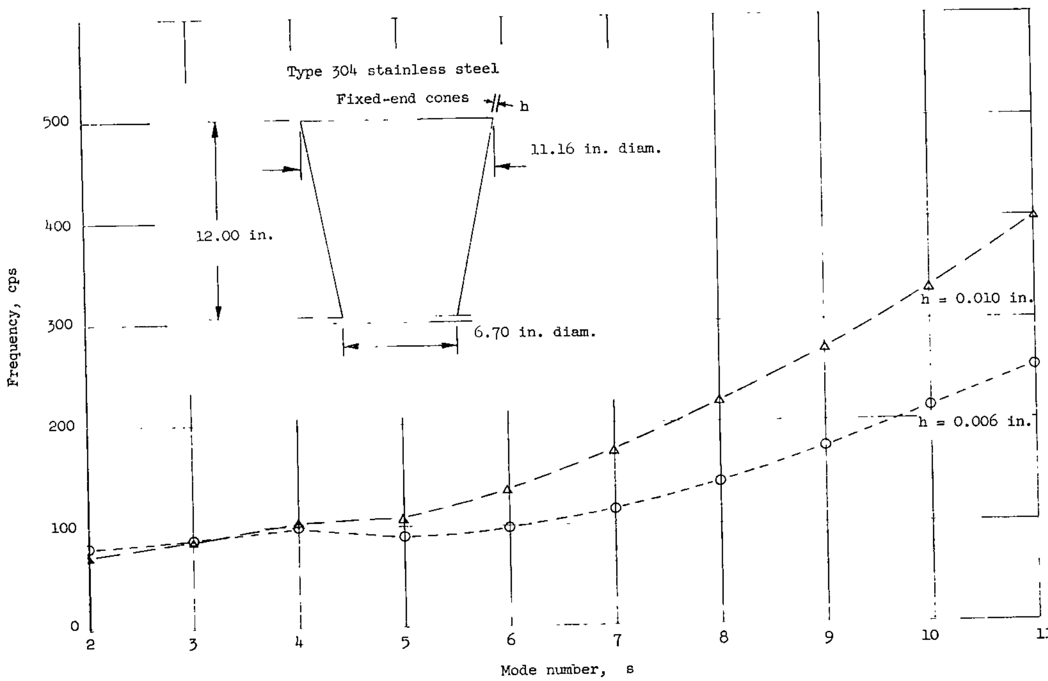


Figure 7.- Experimental frequencies with incomplete end fixity.

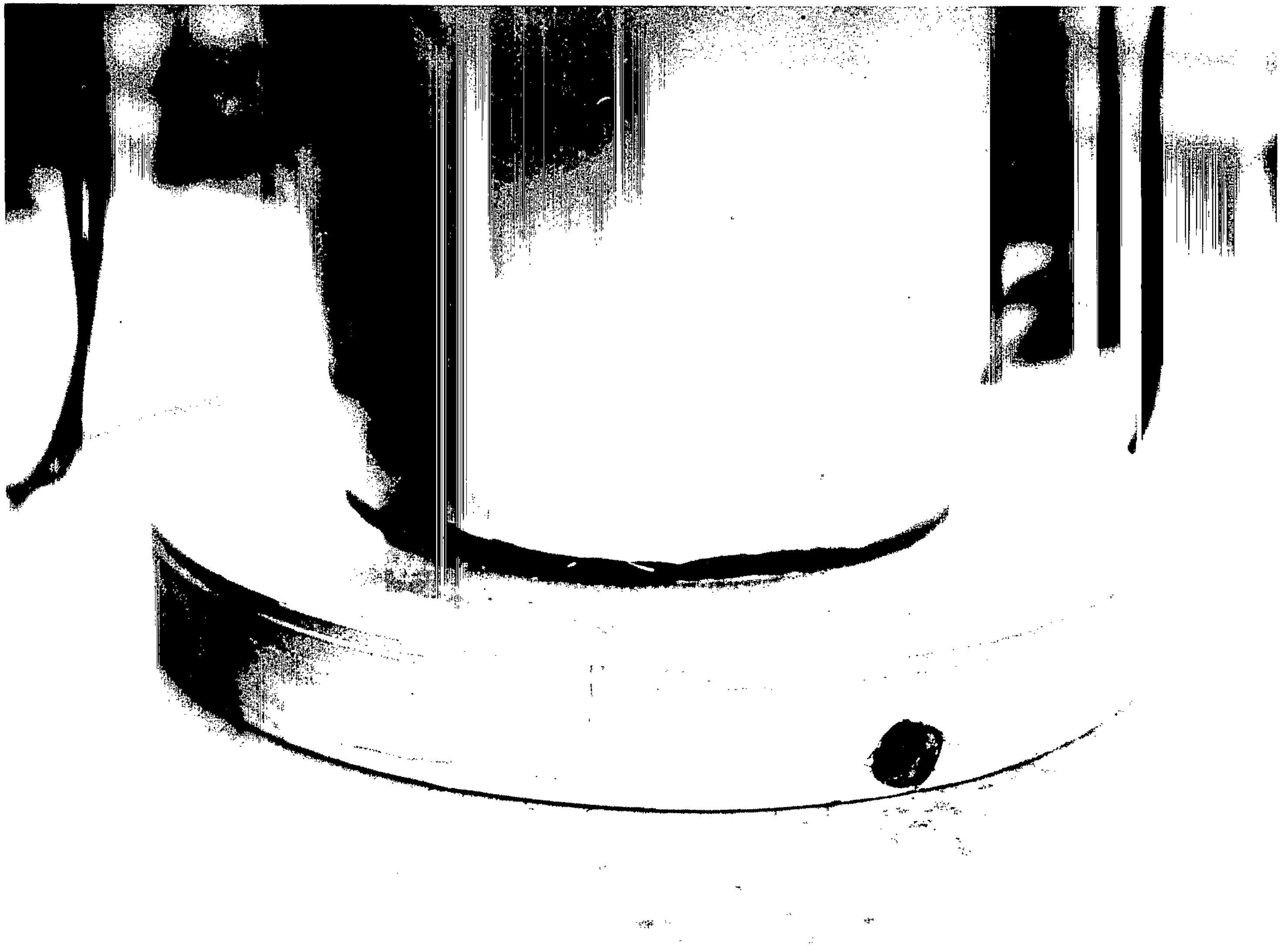


Figure 8.- Conical shell imbedded in low-melting-point bismuth-tin alloy.

L-65-24

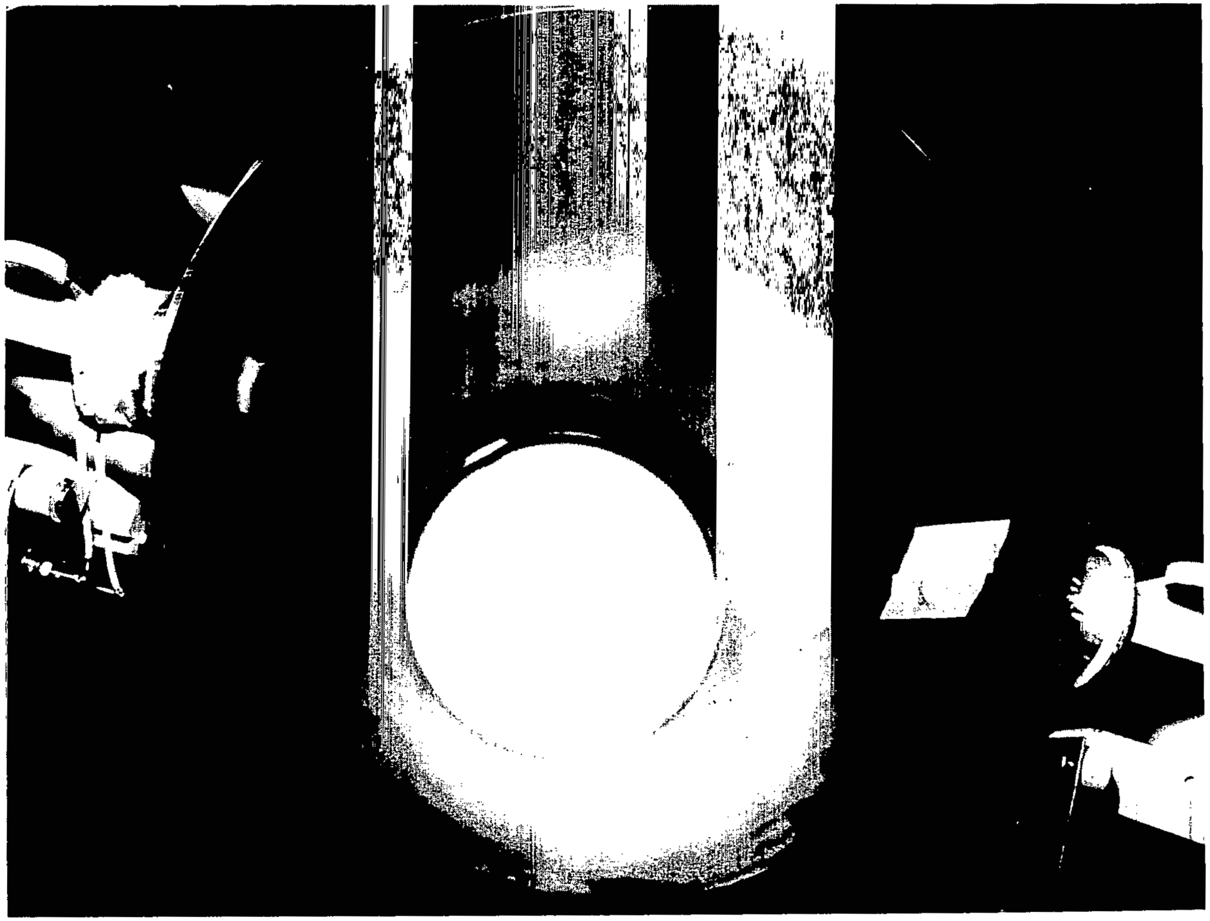


Figure 9.- Vibration nodes for circumferential mode $s = 4$. L-65-25

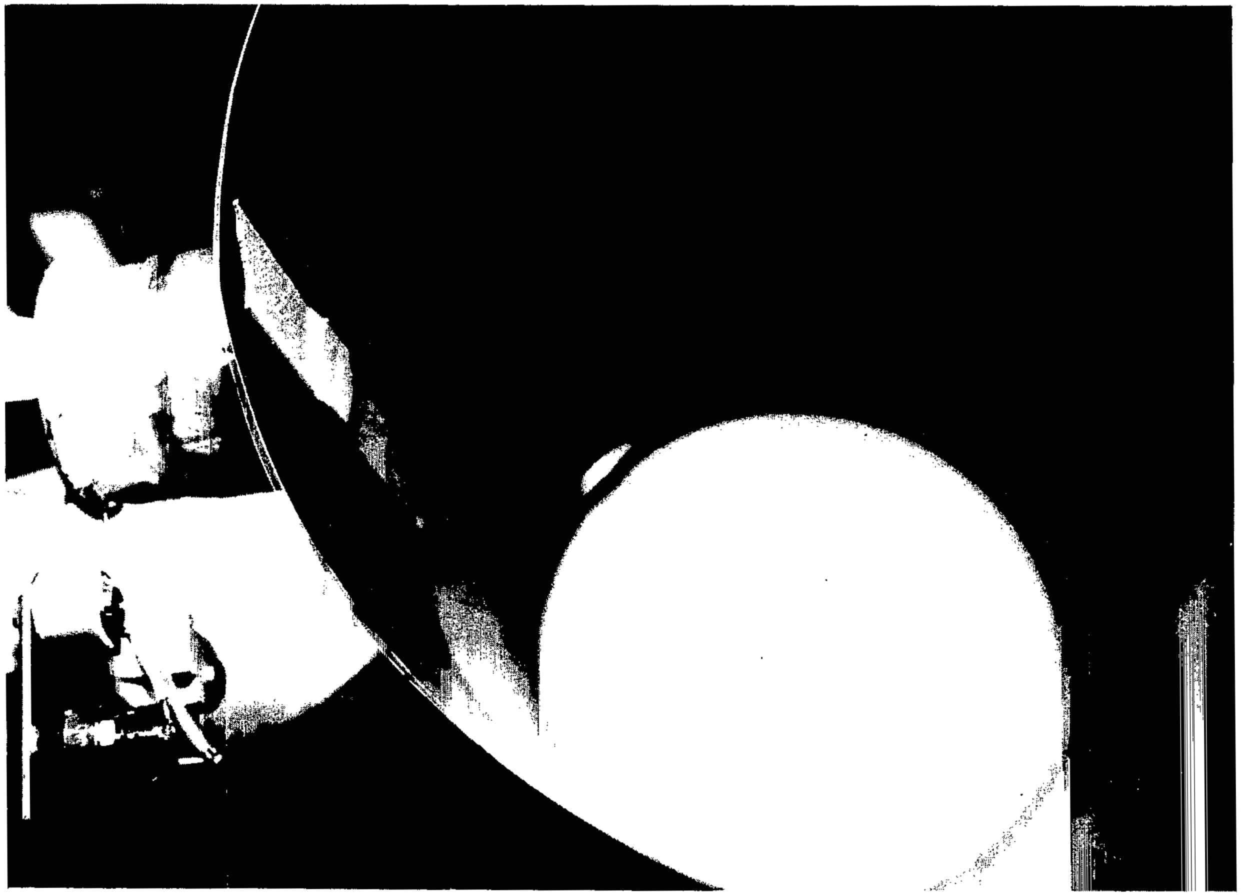


Figure 10.- Closeup of vibration mode for $s = 3$. L-65-26

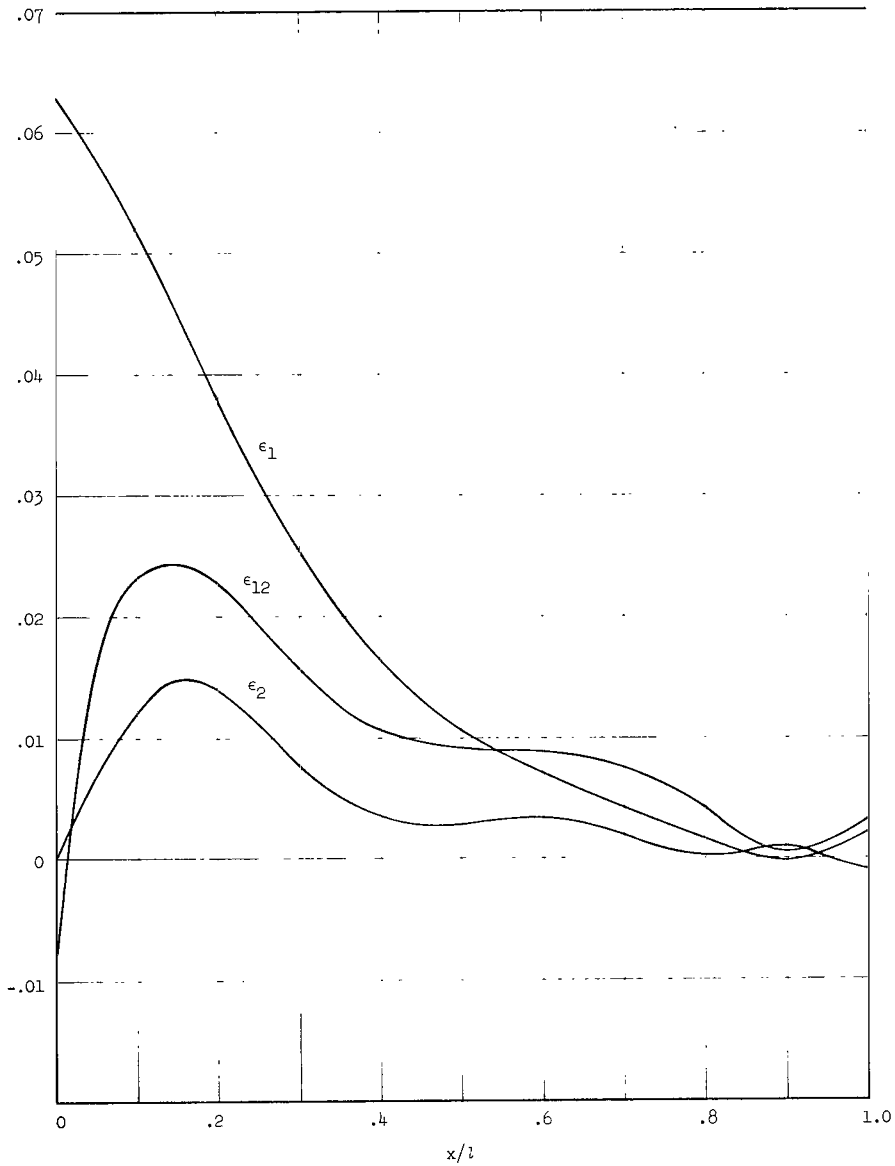


Figure 11.- Comparison of maximum extensional strains for $s = 4$ ($\lambda = 4$, $\alpha = 15^{\circ}$).

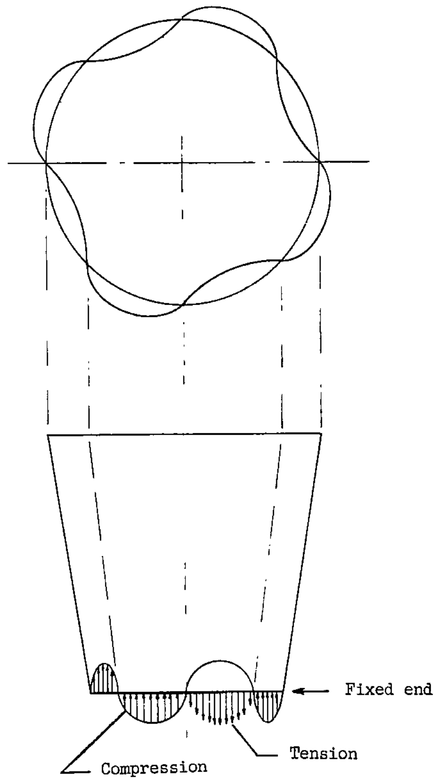


Figure 12.- Axial stress distribution for $s = 4$.

NASA-Langley, 1965

2/22/85

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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